Márk Mezei (Princeton)

MM, Stanford [to appear]; MM [to appear];
Casini, Liu, MM [1509.05044]; Cotler, Hertzberg, MM, Mueller [to appear]

Strings 2016
Entanglement generation and chaos
- Two velocities
- Bounds

Data on entanglement growth
- Holographic results
- Spin chain results

Interpretation and benchmarking
- Operator growth model
- Free streaming, free scalar theory

Summary and open questions
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Summary and open questions
Entanglement generation in global quenches

Global quench:
- Thermalization in a pure state $|\psi(t)\rangle$
- Start with QFT in a short-range entangled state at $t=0$. (E.g. inject uniform energy density or change the Hamiltonian)
- One-point functions reach thermal value $t_{loc} \sim 1/T$
- EE (similarly to $\langle \phi(R) \phi(0) \rangle$) take $t_s \sim R$ to saturate to thermal value
- Good diagnostic of thermalization is how close $\rho_A (|\psi(t)\rangle)$ is to $\text{Tr}_A e^{-\beta(E) H}$

$S_0 = \frac{A \Sigma}{\delta^{d-2}} + \ldots$

Typical point inside is unentangled with outside

$S_{eq} = s_{th} V_A + \ldots$

Typical point inside is entangled with outside
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What is the time evolution of EE?
- 2d: numerics, CFT techniques [Huse, Kim; MM, Stanford; Calabrese, Cardy]
- $d>2$: holography, free field theory [Hartman, Maldacena; Liu, Suh; Cotler, Hertzberg, MM, Mueller]
Monotonicity of relative entropy combined with emergent light cones
- $\nu_B$ cone at finite temperature in chaotic systems
- Monotonicity of relative entropy for subsystems
- Tsunami bound [Afkhami-Jeddi, Hartman]

$$S[A(t)] \leq S[A'(t')] + s_{th} \left( V[A(t)] - V[A'(t')] \right)$$
Bounds on entanglement growth

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Proposed inequality

\[
\partial_t S[A(t)] \leq v_E s_{th} A_\Sigma
\]

- Rigorous versions exist for lattice systems
- Can be proven in holography [MM]

Combination of the two bounds captures many of the essential details of entanglement growth in chaotic systems.
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Holographic models of quenches
- Dual of Cardy-Calabrese boundary state is eternal BH with end of world brane [Hartman, Maldacena]
- Injecting energy density is dual to a collapsing shell. Saturation happens when the HRT surface touches the shell [Liu, Suh]
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- The two setups are equivalent for large R [MM]
- \( v_E \) is determined by behind the horizon physics
- Saturation is determined by near horizon physics, and EE saturates as fast as possible
  \[
  t_S = \frac{R}{v_B}
  \]

Conceptual argument based on entanglement wedge reconstruction.
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Conceptual argument based on entanglement wedge reconstruction.

• Using the NEC, we can show that there are non-trivial constraints on these velocities:

\[ v_E \leq v_E^{(SBH)}, \quad v_B \leq v_B^{(SBH)}, \quad v_E \leq v_B \]
Detailed understanding of how HRT surfaces are behaving

- For large $R$, we can understand the entropy analytically
- In both setups the minimal surfaces are close to a critical surface determined by an **algebraic equation**.
- They shoot out to the boundary exponentially fast.
Detailed understanding of how HRT surfaces are behaving

- For large $R$, we can understand the entropy analytically.
- In both setups the minimal surfaces are close to a critical surface determined by an algebraic equation.
- They shoot out to the boundary exponentially fast.
- Entropy and time are given by the critical surface.

Holographic results on entanglement
Spin chain results on entanglement and chaos

Chaotic spin chain Hamiltonian: \( H = - \sum_i (Z_i Z_{i+1} - 1.05 X_i + 0.5 Z_i) \)

- Entropy growth and \( v_E \):

![Graph showing N = 26 spin chain, A = first 12 sites](image)

![Graph showing dependence of \( v_E \) on Renyi index](image)
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Chaotic spin chain Hamiltonian: 

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- Entropy growth and \( v_E \):

- Operator growth [Roberts, Susskind, Stanford]

\[ v_B = 2.0 > v_E, \quad t_S > \frac{R}{v_B} \]
Comparison with bounds

Combination of the two bounds comes very close to the data from chaotic systems.

- $d=2$: linear growth until saturation
Combination of the two bounds comes very close to the data from chaotic systems.

- **d=2**: linear growth until saturation
- **d>2**: three regimes

Middle regime in good agreement with holographic theories.
Operator growth model

Operator counting model [Abanin, Ho]

- Closer in spirit to spin chains, infinite temperature
- The reduced density matrix is an operator, so it also spreads

\[
\rho(0) = |\uparrow\uparrow\ldots\uparrow\rangle\langle\uparrow\uparrow\ldots\uparrow| = \prod_i \frac{\mathbb{I}_i + Z_i}{2} = \frac{1}{2^{V/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(0)
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- Second Rényi entropy:

$$\text{Tr}_A \rho_A(t)^2 \approx \frac{1}{2^{V_A}} \sum_{\mathcal{O}(0)} \text{Tr}_A \left( \mathcal{O}(t)_A^2 \right)$$

- Small operators contribution: 1
  Big operators: probability of staying inside \( \text{Tr}_A \left( \mathcal{O}(t)_A^2 \right) = 2^{-\alpha s_{th} A[\mathcal{O}(0)](t-t_{\text{delay}})} \)

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- Have to sum over all operators
- Saturates the combined bounds, gives microscopic picture for them

  - \( t_s \) is determined by when the last small operator gets out
  - In the spin chain we can measure \( \alpha \) independently, good agreement with the data for \( S_2(t) \)
Calabrese-Cardy model: energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically. In this model $v_B$ is not captured.

- Leads to linear growth with $v_E = 1$ in 2d.
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Bound on the entanglement velocity from SSA:

$$v_E \leq v_E^{(EPR)} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} < v_E^{(SBH)}$$

Slower than holography.
Free streaming model of entanglement spread

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- In strongly coupled systems, entanglement propagates faster than what’s possible for free particles streaming at the speed of light!

  - $t_S^{(SBH)} > t_S$ is achievable, makes free streaming look even less effective

- Consider the effect of interactions: tensor network picture emerging from scattering particles is natural [Hartman, Maldacena; Casini, Liu, MM]
In a free theory for time dependent Gaussian states the symplectic eigenvalues of the (reduced) correlation matrix determine the entanglement entropy.

- Numerical results for 3d boundary state quench for scalar field [Cotler, Hertzberg, MM, Mueller]
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Open questions

• What is an independent characterization of \( v_E \)?
• Can the bound from relative entropy be saturated in a QFT? Are the holographic bounds \( v_E \leq v_E^{(SBH)}, \ v_B \leq v_B^{(SBH)} \) universal?
• The velocities and \( t_s \) are new observables in a QFT. Are they calculable?
  - What are they in weakly coupled theories? [\( v_B: \text{Stanford} \)]
  - What are they for perturbed 2d CFTs? [\( v_E: \text{Cardy} \)]