Current Algebra Constraints on Supersymmetric Quantum Field Theories

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Main Theme

Conserved charges $Q_i$ generate continuous symmetries. Their (graded) commutators define the symmetry algebra $\mathcal{A}$.

- If the charges $Q_i$ annihilate the vacuum, $Q_i|0\rangle = 0$, then all states lie in representations $\mathcal{R}$ of the symmetry algebra $\mathcal{A}$.
- If $Q_i|0\rangle \neq 0$ the symmetry is spontaneously broken.

In unitary theories $\mathcal{R}$ should be a unitary representation of $\mathcal{A}$.

Natural questions (many examples, long history):

- When can an algebra $\mathcal{A}$ arise as a physical symmetry algebra?
- Which representations $\mathcal{R}$ of a symmetry algebra $\mathcal{A}$ can occur?

We will examine two examples involving supersymmetric QFTs:

- $\mathcal{A} =$ superconformal algebra, $\mathcal{R} =$ local operators
- $\mathcal{A} =$ extended Poincaré SUSY algebra, $\mathcal{R} =$ particles, strings
Current Algebra

In QFT, we expect the generators $Q_i$ of continuous symmetries to arise from local currents $J_i(x)$. Like all well-defined local operators, they should reside in a multiplet $\mathcal{J}$ of the symmetry algebra $\mathcal{A}$.

$$\mathcal{J} \supset \{ J_i(x) \} \quad \longrightarrow \quad Q_i = \int dx \, J_i(x)$$

This talk: current algebra = action of the $Q_i$ on the operators in the current multiplet $\mathcal{J}$, e.g. $Q_i J_j(x)$. Integrating over $x$, we must recover the charge algebra. This is a nontrivial constraint on $\mathcal{J}$, $\mathcal{A}$. 

References: [Weinberg, Witten]
Current Algebra

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This talk: current algebra = action of the $Q_i$ on the operators in the current multiplet $\mathcal{J}$, e.g. $Q_i J_j(x)$. Integrating over $x$, we must recover the charge algebra. This is a nontrivial constraint on $\mathcal{J}, \mathcal{A}$. Some representations of $\mathcal{A}$ may be inconsistent with the existence of local currents. Example [Weinberg, Witten]: If $\mathcal{A} = \text{Poincaré algebra}$, then $\mathcal{J} = T_{\mu\nu}$ is the stress tensor. There are massless single-particle representations of $\mathcal{A}$ for any helicity $h \in \frac{1}{2}\mathbb{Z}$, but

$$\langle p', h | T_{\mu\nu}(q) | p, h \rangle \neq 0 \quad \Rightarrow \quad |h| \leq 1 .$$

In the forward limit $q \to 0$ this measures the energy of the particle (via soft graviton scattering): must be IR finite and nonzero.
Maximal Supersymmetry in QFT

Massless single-particle representations of \( \{Q, Q\} \sim P \) violate the Weinberg-Witten bound \( |h| \leq 1 \) when \( d \geq 4 \) and \( N_Q > 16 \). This leads to the standard lore that QFT requires \( N_Q \leq 16 \).
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- Not true in \( d = 3 \) (no notion of helicity for massless particles), e.g. an \( \mathcal{N} = 9 \) free hypermultiplet exists. It has 16 free bosons \( \phi^i \) and 16 free Majorana fermions \( \psi^i_\alpha \) (\( \mathfrak{so}(9)_R \) spinors).
- Does not rule out interacting SCFTs with \( N_Q > 16 \) (any \( d \)).
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In SCFTs \( A = \) superconformal algebra. Algebraically consistent \( A \)'s are classified [Nahm], very restricted in \( d \geq 3 \):

\[
\begin{align*}
\text{osp}(\mathcal{N}|4) & \quad \text{su}(4|\mathcal{N}) & \quad f(4) & \quad \text{osp}(8|\mathcal{N}) & \quad \text{none} \\
\text{\( d = 3 \)} & \quad \text{\( d = 4 \)} & \quad \text{\( d = 5 \)} & \quad \text{\( d = 6 \)} & \quad \text{\( d \geq 7 \)}
\end{align*}
\]

5d is exceptional (only \( \mathcal{N} = 1 \)), 6d requires chiral \( (\mathcal{N}, 0) \) SUSY. In \( d = 3, 4, 6 \) candidate algebras exist for every \( \mathcal{N} \in \mathbb{Z}_{\geq 0} \).
Maximal Supersymmetry in QFT (cont.)

Not all superconformal algebras $\mathcal{A}$ admit a current algebra interpretation. The required current multiplet $\mathcal{T}$ contains the $R$-symmetry current $R_{ij}^i$, the traceless SUSY current $S_{\mu\alpha}^i$ (gives $Q, S$-supercharges), and the traceless stress tensor $T_{\mu\nu}$ (gives $P_\mu, D, K_\mu$). The commutation relations of $\mathcal{A}$ require that

$$(\star) \quad \mathcal{T} \supset \{ R_{ij}^i, S_{\mu\alpha}^i, T_{\mu\nu} \} , \quad QR \sim S , \quad QS \sim T , \quad QT \sim 0$$

Moreover, $\mathcal{T}$ must be a unitary multiplet of $\mathcal{A}$.
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Moreover, $\mathcal{T}$ must be a unitary multiplet of $\mathcal{A}$.

We have developed a uniform procedure to tabulate the operator content of any unitary superconformal multiplet [Dolan, Osborn;...]. In particular, we analyzed all multiplets with conserved currents:

- If $\mathcal{T}$ exists, it is essentially unique, with a single lowest weight.
- No candidate $\mathcal{T}$ satisfying the constraints (⋆) exists if $d = 4, 6$ and $N_Q > 16$ (talk by [Vafa]).
- In 3d $\mathcal{T}$ exists for any $\mathcal{N}$. If $\mathcal{N} \geq 9$, then $\mathcal{T}$ contains higher-spin currents; the theory is free [Maldacena, Zhiboedov].
Deformations of SCFTs

Our machinery also leads to a classification of all possible SUSY deformations of SCFTs by local operators. Many applications, e.g. universal constraints on SUSY RG-flows. Example: 4d $\mathcal{N} = 2$ SCFTs [Argyres et. al.], $SU(2)_R \times U(1)_r$ symmetry ($r(Q^i_\alpha) = -1$).
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Only two kinds of Lorentz-scalar relevant or marginal deformations:

- A flavor current resides in a real multiplet $\mathcal{J}^{(ij)}$ such that
  \[ Q^i_\alpha \mathcal{J}^{jk} = \mathcal{Q}^i_\beta \mathcal{J}^{jk} = 0 \, , \, \Delta \mathcal{J} = 2 \, , \, \sigma^{i\mu} j_\mu \sim Q^i_\alpha \mathcal{Q}^j_\beta \mathcal{J}_{ij} \, . \]

  \[ \Delta L = (Q^2)^{ij} \mathcal{J}_{ij} \text{ preserves SUSY}, \, SU(2)_R, \text{ breaks } U(1)_r. \]

- Chiral operators satisfy $\mathcal{Q}^i_\alpha \mathcal{O} = 0$ and $\Delta \mathcal{O} = r > 1$.

  \[ \Delta L = Q^4 \mathcal{O} \text{ preserves SUSY}, \, SU(2)_R, \text{ typically breaks } U(1)_r. \]

The upshot is that all deformed SCFTs have an $SU(2)_R$ symmetry, but generically not $U(1)_r$ (the same conclusion applies to gauging). If there is a Coulomb branch, then $SU(2)_R$ is unbroken there.
Non-Conformal 4d $\mathcal{N} = 2$ Theories

Now $\mathcal{A} = \text{Poincaré SUSY algebra} \rtimes SU(2)_R$. It can be extended by $p$-form charges carried by $p$-brane excitations:

$$\{ Q^i_\alpha, \overline{Q}_{j\beta} \} = 2\sigma^\mu_{\alpha\beta} \left( \delta^i_j P_\mu + (X_\mu)^i_j \right),$$

$$\{ Q^i_\alpha, Q^j_\beta \} = 2\sigma^{\mu\nu}_{\alpha\beta} Y_{[\mu\nu]}^{(ij)} + 2\varepsilon_{\alpha\beta} \varepsilon^{ij} Z,$$

$$[R^{(ij)}, Q^k_\alpha] = -\varepsilon^k(i Q^j_\alpha).$$

The charged states are **strings** for $(X_\mu)^i_j$, **domain walls** for $Y_{[\mu\nu]}^{(ij)}$, and **particles** for $Z$. Unitarity, with $(Q^i_\alpha)^\dagger = \overline{Q}_{i\dot{\alpha}}$, implies a BPS bound for their mass (or tension):

$$M_{\text{string}} \geq |X|, \quad M_{\text{domain wall}} \geq |Y|, \quad M_{\text{particle}} \geq |Z|.$$

When this bound is saturated, we can get BPS strings, domain walls, or particles. Which of these excitations can arise in $\mathcal{N} = 2$ QFTs, and what can we say about their quantum numbers?
$\mathcal{N} = 2$ Stress-Tensor Multiplets

The current multiplet $\mathcal{T}$ that gives rise to the SUSY algebra $\mathcal{A}$ is again the stress-tensor multiplet. All charges arise from currents:

$$(\star) \quad \mathcal{T} \supset \{ R^{(ij)}_\mu, S^i_\mu\alpha, T_\mu\nu, (x_{\mu\nu})^i_j, (y_{\mu\nu\rho})^{(ij)}, z_\mu \}$$

Now $T_\mu\nu, S_\mu\alpha$ are not traceless. The charge algebra $\mathcal{A}$ fixes

$$(\dagger) \quad \overline{Q}S \sim T + x, \quad QS \sim y + z, \quad Q(T, x, y, z) \sim 0, \quad QR \sim S$$
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Qualitative differences with the stress-tensor multiplet in SCFTs:

- $\mathcal{A}$ may admit distinct representations satisfying $(\star)$, $(\dagger)$.
- A given theory may have two (or more) multiplets $\mathcal{T}, \mathcal{T}'$. Then $T_{\mu\nu}$, $T'_{\mu\nu}$ and $S_{\mu\alpha}$, $S'_{\mu\alpha}$ differ by improvement terms.
- The other currents in $\mathcal{T}$, $\mathcal{T}'$ need not differ by improvements.

Example: $R_{\mu}^{(ij)}$ can mix with an $SU(2)$ flavor current.

A complete list of possible $\mathcal{N} = 2$ stress-tensor multiplets is not available, but we know several examples. Is there a preferred one?
The Sohnius Stress-Tensor Multiplet

Nearly all non-conformal $\mathcal{N} = 2$ theories with $SU(2)_R$ symmetry seem to admit a stress-tensor multiplet $\mathcal{T}$ introduced by [Sohnius]:

$$(\mathcal{T})^\dagger = \mathcal{T} , \quad \varepsilon^{\alpha\beta} Q^{(i}_\alpha Q^{j)}_\beta \mathcal{T} = Z^{(ij)} , \quad Q^{(i}_\alpha Z^{jk)}_\beta = \overline{Q^{(i}_\alpha Z^{jk)}_\beta} = 0 .$$

- $Z^{(ij)}$ is a complex flavor current multiplet that contains $z^\mu$. When it vanishes, we recover the superconformal multiplet.
- An $\mathcal{N} = 2$ version of the $\mathcal{N} = 1$ multiplet [Ferrara, Zumino].
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$$ T \rightarrow \psi^i_\alpha \rightarrow \mathcal{Z}^{(ij)}, W_{[\mu\nu]}, R^{(ij)}_\mu, r_\mu \rightarrow S^i_{\mu\alpha}, \chi^i_\alpha \rightarrow T_{\mu\nu}, z_\mu, C $$

- $W_{[\mu\nu]}, r_\mu$ are not conserved. SCFT: $r_\mu = U(1)_r$ current.
- There are no genuine currents $(x_{[\mu\nu]})^i_j$ or $y^{(ij)}_{[\mu\nu\rho]}$. Hence there are no BPS strings or domain walls.
- Consistent with $\mathcal{N} = 1$ [TD, Seiberg]: no BPS strings with an [FZ]-multiplet, no BPS domain walls with an $R$-symmetry.
BPS Particles in 4d $\mathcal{N} = 2$ Theories

Typically studied on Coulomb branch, where $SU(2)_R$ is unbroken.

$$\{ Q^i_\alpha, S^j_{\mu\beta} \} = 2 \varepsilon^{ij} \varepsilon_{\alpha\beta} \left( z_\mu + \partial^\nu W^+_{[\mu\nu]} \right), \quad W^+_{[\mu\nu]} \sim F^+_{[\mu\nu]} \quad \text{[Witten, Olive]}$$

Pick a vacuum and charge sector. Then $Z \in \mathbb{C}$ is fixed and can be aligned with $\mathbb{R}$: particles have $Z > 0$, antiparticles have $Z < 0$. In the rest frame $P^\mu = (M, 0)$, little group is $SU(2)_J \times SU(2)_R$.

$$A^{(\pm)i}_\alpha = Q^i_\alpha \pm \sigma^0_{\alpha\beta} \dot{Q}^{i\beta}, \quad \left( A^{(\pm)i}_\alpha \right)^\dagger = \pm A^{(\pm)i}_i$$

$$\left\{ A^{(\pm)i}_\alpha, A_{(\mp)j}^\beta \right\} = 0, \quad \left\{ A^{(\pm)i}_\alpha, A^{(\pm)j}_\beta \right\} = 4 \varepsilon^{ij} \varepsilon_{\alpha\beta} (Z \pm M)$$

- BPS particles satisfy $M = Z > 0$, and hence $A^{(-)} = 0$. Four states in a half hypermultiplet: $|\uparrow\rangle \leftrightarrow A^{(+)}(+) \leftrightarrow |i = 1, 2\rangle \leftrightarrow A^{(+)} \leftrightarrow |\downarrow\rangle$.
- Anti-BPS particles: $M = -Z > 0$, roles of $A^{(\pm)}$ are reversed.
- Long multiplets: $M > |Z|$, $A^{(\pm)} \neq 0$. This leads to 16 states.
The NEC and its Consequences

More generally, we can tensor the half hypermultiplet with any representation \((j; r)\) of the \(SU(2)_J \times SU(2)_R\) little group:

\[
\left| \uparrow; m = -j, \ldots, j, \ s = -r, \ldots, r \right> \xrightarrow{A^{(+)}} \left| i = 1, 2; \ m, s \right>
\]

- Multiplets with \(j \neq 0\) occur, e.g. \((j = \frac{1}{2}; r = 0)\) is a W-boson.
- Empirically, multiplets with \(r \neq 0\) do not seem to occur in QFT. This was formalized in the no-exotics conjecture (NEC) of [Gaiotto, Moore, Neitzke]. Putative multiplets with \(r \neq 0\) are called exotic. Further work by [Diaconescu et.al.; del Zotto, Sen].

The conjecture has implications for physics and mathematics:
- Long multiplets cannot hit the BPS bound and decay into short ones, because some fragments would have to be exotic.
- Protected indices, which count BPS states with signs, actually coincide with the physical degeneracies (cf. BH microstates).
- Implies constraints on the cohomology of moduli spaces that arise in counting BPS states [Moore, Royston, van Den Bleeken].
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Flavor Symmetries and Mixing

In the presence of an $SU(2)_{\text{flavor}}$ symmetry, the $SU(2)_R$ symmetry is not unique: $\widehat{SU}(2)_R = SU(2)_R \times SU(2)_{\text{flavor}}|_{\text{diag}}$ is just as good.

Example: massless hypermultiplet $q^{i,a}, \psi^a_\alpha$. Here $i$ an $R$-symmetry doublet index, and $a$ is a flavor doublet index.
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Mixing: $i \rightarrow \tilde{i}, a \rightarrow \tilde{j}$, where $\tilde{i}, \tilde{j}$ are $\widetilde{SU}(2)_R$ doublet indices.

Therefore the hypermultiplet is exotic with respect to $\widetilde{SU}(2)_R$.

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- We would like to state the NEC with respect to $SU(2)_R$. How do we distinguish it in a model-independent way?
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While the two $R$-symmetries are indistinguishable at the level of charges, they arise from different current algebras. The current $R^{(ij)}_{\mu}$ resides in the Sohnius stress-tensor multiplet, while $\tilde{R}^{(ij)}_{\mu}$ resides in a structurally different, less familiar multiplet.
Current-Algebra Proof of the NEC

Goal: prove the NEC with respect to the $SU(2)_R$ current $R^{(ij)}_{\mu}$ in the Sohnius multiplet. We will examine its forward matrix elements between BPS states, where it measures the charges $R^{(ij)}$. For now, we assume that all forward limits exist, postponing a small subtlety.

Argue by contradiction: consider $\langle \uparrow; s | R_{22}^0 | \uparrow; s' \rangle$. If $r \neq 0$ (exotic), choose $s = r - 1, s' = r$ to get a nonzero matrix element for the lowering operator $R_{22}^0$. Claim: in fact, it actually vanishes.

Sohnius multiplet: $R_{22}^0 \sim Q Q^T$. BPS state: $A(\bar{a}) \sim Q - Q = 0$.

We can move the $Q$'s around to derive a Ward identity:

$$\langle \uparrow; r - 1 | R_{22}^0 | \uparrow; r \rangle = -2M \langle 1; r - 1 | T | 2; r \rangle$$

There are many other such Ward identities (interesting), but they are not sufficient to show that the matrix element vanishes.
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Extra tool: the $\Theta = \text{CPT}$ symmetry of relativistic QFT.
Since $\Theta^2 = (-1)^F$, the SUSY algebra determines (up to a sign)
$$\Theta Q^i_\alpha \Theta^{-1} = i\overline{Q}_{i\dot{\alpha}}, \quad \theta Z \theta^{-1} = -\overline{Z}.$$ 
This fixes the $\Theta$-transformations of all Sohnius multiplet operators.
Current-Algebra Proof of the NEC (cont.)

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\[ \langle \uparrow; r - 1 | R_0^{22} | \uparrow; r \rangle \xrightarrow{\text{Ward ID}} -2M \langle 1; r - 1 | T | 2; r \rangle \]

\[ \Theta R_0^{22} \Theta^{-1} = R_0^{11} \]

\[ + \Theta \langle \uparrow; r - 1 | R_0^{11} | \uparrow; r \rangle \Theta = \]

\[ = - \Theta \langle \uparrow; r - 1 | R_0^{11} | \uparrow; r \rangle \Theta \]

\[ \xrightarrow{\text{Ward ID}} -2M \Theta \langle 1; r - 1 | T | 2; r \rangle \Theta \]
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Subtlety: forward matrix elements of $\mathcal{T}$ are divergent, due to soft single-photon exchange. This IR effect can be computed exactly and subtracted: $\mathcal{Z}^{(ij)} \rightarrow \mathcal{Z}^{(ij)}_{\text{eff}}$ (E&M boundary terms in $Z$).
Conclusions and Extensions

- General lesson (not new): in QFT, current algebra can exclude phenomena that are allowed at the level of the charge algebra.
- Two examples:
  - SCFTs with $N_Q > 16$ in $d \geq 3$ (in $d = 3$, interacting SCFTs).
  - Exotic BPS states in 4d $\mathcal{N} = 2$ theories [GMN]
- The argument against exotics did not require a UV-complete theory. Consider a 5d $\mathcal{N} = 1$ QFT with a Sohnius multiplet, compactified on $S^1$. Some 4d BPS states come from BPS strings wrapping $S^1$, so the strings cannot carry $R$-charge.
- The discussion can be repeated for BPS particles in 5d $\mathcal{N} = 1$ theories and BPS strings in 6d $(1, 0)$ theories. There is also a 3d version (richer due to $SU(2)_R \times SU(2)'_R$ symmetry).
- It would be interesting to extend the argument to framed BPS states, which are bound to a BPS defect.

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