Scattering via Riemann spheres

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based on works with Freddy Cachazo & Ellis Yuan
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S-matrix in QFT

- **Colliders at high energies** need amplitudes of e.g. many gluons (tree & loop level)

  \[ gg \rightarrow gg \ldots g \]

- **Fundamental level**: understanding of QFT incomplete; tensions with gravity
  new structures & simplicity seen in (perturbative) scattering amplitudes

- **Goal**: deeper understanding of QFT & gravity from studying the S-matrix
Surprising simplicity

- theoretical challenges: many diagrams, many many terms, gauge (non-)invariance

\[ n \text{-gluon scattering (tree)} \]

<table>
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<th>6</th>
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- Why? redundancies in textbook formulation of QFT, but unexpected simplicity & structures emerge for on-shell S-matrix, e.g. MHV gluon amplitudes [Parke, Taylor, 86]

\[ M_n(i^-, j^-) = \frac{\langle i \, j \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \cdots \langle n \, 1 \rangle}, \quad k^\mu = (\sigma_\mu)_{\alpha, \bar{\alpha}} \lambda^\alpha \bar{\lambda}^{\bar{\alpha}}, \quad \epsilon^\mu = \ldots \]

\[ \langle a \, b \rangle := \varepsilon_{\alpha, \beta} \lambda^\alpha_a \lambda^\beta_b, \quad [a \, b] := \varepsilon_{\bar{\alpha}, \bar{\beta}} \bar{\lambda}^\bar{\alpha}_a \bar{\lambda}^{\bar{\beta}}_b \]

[Xu, Zhang, Chang, 84...]

- Led to 30 years of enormous progress on computing & understanding S-matrix!
Twistor-string revolution

- **Witten’s twistor string theory** → worldsheet model for gluon tree amplitudes
  
  amps = string correlators with a map from $\mathbb{CP}^1$ to $\mathbb{CP}^{3|4}$ (twistor space)  
  [Witten, 2003]

- **Key observation**: [Nair, 88] Parke-Taylor MHV amps = correlator on $\mathbb{CP}^1$

\[ \lambda_i^\alpha \sim (z_i, 1), \quad PT_n := \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)} \cdot j_A(z)j_B(z') = \frac{f_{AB}^{C}}{z - z'} + \text{double poles} + \ldots \]

- $N^k$ MHV amplitude is the image of $PT_n$ under polynomial map of deg. $(k+1)$; polarization dependence naturally encoded by maximal supersymmetry.

- Inspired CSW & BCFW, progress on unitarity method, Grassmannian, etc. etc.
Cachazo-He-Yuan formulation

- Witten’s twistor string very special: d=4 N=4 super Yang-Mills theory
  - no supersymmetry? any spacetime dimension?
  - general theories: gravity, Yang-Mills, standard model, effective field theories?
  - generalizations to loop level?

- **CHY formulation**: scattering of massless particles in any dimension [CHY 2013]
  - compact formulas for amplitudes of gluons, gravitons, fermions, scalars, etc.
  - manifest gauge (diff) invariance, double-copy relations, soft theorems, etc.
  - string-theory origin: “ambitwistors”/“chiral” strings [Mason, Skinner; Adamo et al; Berkovits; Siegel] [c.f. Yu-tin’s talk]
Scattering equations

\[ E_a := \sum_{b=1,b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a = 1, 2, \ldots, n \]

- universal, independent of theories; determine locations of punctures in terms of kinematics
- physical singularities ↔ boundary of moduli space for \( n \)-punctured Riemann spheres
- simplest “derivation”: saddle point eqs in tensionless limit of string amps [Gross, Mende]

\[ E_a = \frac{\partial [\Sigma_{i \neq j} s_{i,j} \ln(\sigma_{i,j})]}{\partial \sigma_a} \]

\( SL(2, \mathbb{C}) \) symmetry
CHY representation of tree amps

\[ M_n = \int \frac{d^n \sigma}{\text{vol } \text{SL}(2, \mathbb{C})} \prod_a \delta(E_a) \mathcal{I} \{ k, \epsilon, \sigma \} = \sum_{\{ \sigma \} \in \text{solns.}} \frac{\mathcal{I} \{ k, \epsilon, \sigma \}}{J \{ \sigma \}} \]

- S-matrix = localized integral = sum over solutions, of certain CHY integrand
  n-3 integrals/delta functions, (n-3)! solutions \[ \text{[CHY; Dolan, Goddard]} \rightarrow (n-3)! \] “virtual amps”

- Key: a worldsheet picture of massless scattering via n-punctured Riemann spheres. Feynman diagrams and Lagrangians become emergent!

- Task: find “dynamic part”, i.e. CHY integrands for different massless QFT’s
Further developments [not covered in this talk...]

• **Origins from worldsheet**: ambitwistor strings, chiral strings & null strings

\[
\begin{array}{c}
\text{String theory} \xrightarrow{\alpha' \to \infty} \text{Null strings} \\
\downarrow \hfill \downarrow \\
\text{Quantum strings} \xrightarrow{\text{GM}} \text{Higher-spin like} \xrightarrow{x} \text{Ambitwistor strings}
\end{array}
\]


• **Mathematical aspects**: solving scattering eqs, analytical evaluations etc.

• **Extensions & applications**: more QFT & string amps, soft theorems, amp relations
CHY formulas: $\phi^3$, YM & GR

- All tree amplitudes in bi-adjoint $\phi^3$ scalar, Yang-Mills and gravity in any dim [CHY 13]

$$m[\pi|\rho] := \int d\mu_n \; PT[\pi] \; PT[\rho]$$

$$\mathcal{L}_{\phi^3} = -\frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{3!} f^{IJK} f^{I'} J' K' \phi^{I'} \phi^{J'} \phi^{K'}$$

$$M_{n}^{YM}[\pi] = \int d\mu_n \; PT[\pi] \; Pf' \Psi$$

$$M_{n}^{h+B+\phi} = \int d\mu_n \; Pf' \Psi(\epsilon) \; Pf' \Psi(\epsilon') \rightarrow M_{n}^{GR} = \int d\mu_n \; \det' \Psi(\epsilon)$$

- Two ingredients: Parke-Taylor factor (color) & a Pfaffian (polarization): $PT^{2-S} \times Pf^{S}$

$$PT[\pi] := \frac{1}{(\sigma_{\pi(1)} - \sigma_{\pi(2)}) (\sigma_{\pi(2)} - \sigma_{\pi(3)}) \cdots (\sigma_{\pi(n)} - \sigma_{\pi(1)})}$$

$$Pf' \Psi(\sigma, k, \epsilon)$$
The Pfaffian

• The (reduced) Pfaffian of a $2n \times 2n$ skew matrix $\Psi$, with four blocks

$$\text{Pf'}\Psi := \frac{\text{Pf}|\Psi|_{i,j}}{\sigma_{i,j}} \quad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

• Simplified open superstring correlator: $\text{Pf'}\Psi \sim \langle V^{(0)}(\sigma_1) \ldots V^{(-1)}(\sigma_i) \ldots V^{(-1)}(\sigma_j) \ldots V^{(0)}(\sigma_n) \rangle$

• The Pfaffian is permutation invariant, multi-linear in polarizations ...
  most importantly **gauge invariant** on the support of scattering equations!
Gauge & diffeomorphism invariance

• Pf' (det') $\Psi$ as the simplest gauge (diffeo.) invariant object: $\epsilon_{a}^{\mu} \sim \epsilon_{a}^{\mu} + \alpha k_{a}^{\mu}$

\[
\begin{pmatrix}
0 & \ldots & \sum_{b=2}^{n} \frac{k_{b}}{\sigma_{1,b}} \frac{k_{b}}{\sigma_{1,b}} & \ldots \\
\frac{k_{2}}{\sigma_{2,1}} & \ldots & \frac{k_{2}}{\sigma_{2,1}} & \ldots \\
\vdots & \ddots & \vdots & \ddots \\
\frac{k_{n}}{\sigma_{2,1}} & \ldots & \frac{k_{n}}{\sigma_{2,1}} & \ldots \\
-\sum_{b=2}^{n} \frac{\epsilon_{2}}{\sigma_{2,1}} \frac{k_{b}}{\sigma_{1,b}} & \ldots & 0 & \ldots \\
\frac{\epsilon_{n}}{\sigma_{2,1}} & \ldots & \frac{\epsilon_{n}}{\sigma_{2,1}} & \ldots \\
\end{pmatrix}
\]

• It vanishes by scattering equations
• invariant for (n-3)! virtual amplitudes
• closed-string=(open-string)^2
• CHY integrand: $GR = YM^2 / \phi^3$
Double-copy relations

• First such relations discovered as (FT limit of) Kawai-Lewellen-Tye relations:

\[ M_n^{\text{closed}} = \sum_{\alpha,\beta} M_n^{\text{open}}[\alpha] \, S^{\text{string}}[\alpha|\beta] \, M_n^{\text{open}}[\beta] \implies M_n^{\text{GR}} = \sum_{\alpha,\beta} M_n^{YM}[\alpha] \, S[\alpha|\beta] \, M_n^{YM}[\beta]. \]

• How about \( GR = YM^2 / \phi^3 \)? KLT derived from inserting two PT’s in CHY: \( S = m^{-1}. \)

\[ M_n = \int d\mu_n \, I_L \, I_R \implies M_n = \sum_{\alpha,\beta} M_L[\alpha] \, m^{-1}[\alpha|\beta] \, M_R[\beta], \text{ for } M_{L(R)} := \int d\mu_n \, \text{PT} \, I_{L(R)}. \]

• A general way of seeing double-copy relations: splitting a CHY formula into two.
More theories

• Generate CHY formulas of new theories from old ones, e.g. dim reduction
  \( \text{GR} \to \text{Einstein-Maxwell (EM)}, \ \text{YM} \to \text{YM-scalar (YMs)}, \) with Pfaffian factorizes:

\[
M_{n\gamma}^{\text{EM}} = \int d\mu_n \text{Pf}^\prime A \text{Pf} X \text{Pf}^\prime \Psi, \quad M_{n \sigma}^{\text{YMs}} = \int d\mu_n \text{Pf}^\prime A \text{Pf} X \text{PT}; \quad X_{ab} = \frac{\delta^{I_a I_b}}{\sigma_{a,b}}(1 - \delta_{a,b}).
\]

• A new operation to add non-abelian interactions leads to “direct sum” of theories
  Formulas in \( \text{Einstein} \oplus \text{Yang-Mills} \) and \( \text{YM} \oplus \text{bi-adjoint scalar} \) theories [CHY 14]

• A new class: “exceptional” effective field theories (EEFT) with Goldstone scalars
  very special theories: amplitudes have enhanced “Adler’s zero”! [Cheung et al 14] [CHY 14]
More theories

• $M_n = \int d\mu_n (Pf'A)^2 \, PT$, adjoint scalars with two derivative coupling?

  U(N) NLSM (the chiral Lagrangian) \[ \mathcal{L} = \text{Tr}(\partial_\mu U^+ \partial^\mu U) \]

• $M_n = \int d\mu_n (Pf'A)^2 \, Pf'\Psi$, higher-derivative-coupled photons?

  Born-Infeld theory (BI) & DBI by dim reduction \[ \mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu \phi \partial_\nu \phi)} \]

• a special Galileon (single scalar with many derivatives) \[ M_n^{sGal} = \int d\mu_n (Pf'A)^4 \]

• double-copy relations: BI $\sim$ YM $\otimes$ NLSM, DBI $\sim$ YMs $\otimes$ NLSM, $sGal \sim NLSM^2$
A landscape of massless theories
Soft theorems in CHY

- CHY makes manifest old & new soft theorems; connections to BMS etc. [Strominger,...]

\[ M_n^{\text{gauge}} = (S^{(0)} + S^{(1)}) M_{n-1}^{\text{gauge}} + O(\tau), \quad M_n^{\text{GR}} = (S^{(0)} + S^{(1)} + S^{(2)}) M_{n-1}^{\text{GR}} + O(\tau^2), \]

- EEFT’s are the only scalar EFT’s with vanishing soft behavior as \( O(\tau^p) \) for \( p=1,2,3 \)
  Non-linearly realized symmetry: coset from double-soft-scalar theorems [CHY 15]

\[ M_n^{\text{NLSM}} = (S^{(0)} + S^{(1)}) M_{n-2}^{\text{NLSM}} + O(\tau^2), \quad M_n^{\text{DBI}} = (S^{(0)} + S^{(1)} + S^{(2)}) M_{n-2}^{\text{DBI}} + O(\tau^4), \]

(also for sGal). Striking similarities with gauge/gravity soft theorems. Why?

- 4d: manifest double soft theorems in N=4 SYM, N=8 SUGRA & DBI-Volkov-Akulov
  e.g. double fermions: non-linearly realized SUSY [Huang et al 14] \( \leftrightarrow 16+16 \) [Bergshoeff et al 14]
Loops from trees

• Feynman’s loop-tree theorem → loop amps from (generally divergent) forward limits of trees; need off-shell momenta \((l^2 \neq 0)\) & regularizations; e.g.

\[
M_{n}^{1}\text{-loop} \sim \int \frac{d^D l}{l^2} \sum_{l_+ = l_-, \epsilon_+ = (\epsilon_-)^*} M_{n+2}^{\text{tree}}( \{(k_i; 0)\}, \pm (\ell, |\ell|) ),
\]

• Both resolved in CHY: loop-level eqs & formulas on a sphere [Geyer et al 15][HY, CHY 15]

\[
M_{n}^{(1)} = \int d^D l \frac{1}{l^2} \int d\mu_{n}^{(1)} I_n(\{\sigma, k, \epsilon\}; \ell), \quad \mathcal{E}_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} + \frac{k_a \cdot \ell}{\sigma_a}, \quad \text{for } a = 1, \ldots, n.
\]

• Agree with ambitwistor results [Geyer et al 15]; two-loop progress [Geyer et al 16; Feng 16].

• Seems to give wrong propagators \(1/(l + P)^2 - l^2\), but equivalent to standard rep (difference integrates to zero) → a new rep of loop integrands [c.f. Baadsgaard et al 15].
Loops from trees

• Suming over colors/polarizations gives 1-loop “PT” & “Pfaffians” [Geyer et al 15][HY, CHY 15]

\[ PT_n^{(1)}[1, 2, \ldots, n] := \sum_{i=1}^{n} PT_{n+2}[1, \ldots, i, +, -, i+1, \ldots, n]. \]

\[ Pf_s^{(1)} = \frac{1}{\sigma^2_{+,-}} Pf\Psi_n(\ell), \quad Pf_g^{(1)} = \sum_{\epsilon_+=(\epsilon_-)^*} Pf'\Psi_{n+2}(\ell), \quad Pf_f^{(1)} = \ldots \]

• One-loop formula for \( \phi^3 \), YM & GR, also susy-theories by adding fermions [Geyer et al 15]

\[ \mathcal{I}_n^{\phi^3} = (PT_n^{(1)})^2, \quad \mathcal{I}_n^{YM} = PT_n^{(1)} Pf_g^{(1)}, \quad \mathcal{I}_n^{GR} = (Pf_g^{(1)})^2 - c_d (Pf_f^{(1)})^2, \]

\[ \mathcal{I}_n^{SYM} = PT_n^{(1)} (Pf_g^{(1)} - c_d Pf_f^{(1)}), \quad \mathcal{I}_n^{SUGRA} = (Pf_g^{(1)} - c_d Pf_f^{(1)})^2. \]

• Gauge invariance, soft theorems, unitarity cuts, SUSY etc. manifest, but how to systematically understand loop amplitudes/integrals in this new rep?
Back to four dimensions

- CHY rep. simplifies in 4d → old & new “connected” formulas [RSV 04; Cachazo, Skinner 12; SH et al 16]
- **Key**: scattering eqs split into RSV-Witten eqs for sectors, k=2,3...,n-2 [CHY 13; SH et al 16]

A prior nothing to do with helicities, any amp decomposes into n-3 sectors:

\[M_n = \sum_{k=2}^{n-2} \sum_{\text{soln. } k} \frac{I_n^{\text{CHY}}}{J_n} := \sum_{k=2}^{n-2} M_{n,k}, \quad \Rightarrow \quad M_{n,k} = \int d\mu_{n,k}^{4d} I_{n,k}^{4d}.\]

- YM & GR: for k neg. hel., Pf'Ψ'=0 for any soln. sector \( k' \neq k \); Pf' A=0 for \( k' \neq \frac{n}{2} \)
  only right sector needed; 4d formulas simplify a lot & natural to have SUSY!
  
  e.g. 4d integrands in SYM, DBIVA & SUGRA (w. susy-measures): [Geyer et al 14; SH et al 16]

\[\mathcal{I}_{n,k}^{\text{SYM,4d}} = PT_n, \quad \mathcal{I}_{n,\frac{n}{2}}^{\text{DBIVA,4d}} = \det' A_n, \quad \mathcal{I}_{n,k}^{\text{SUGRA,4d}} = \det' H_k \det' \tilde{H}_{n-k}.\]
Fermions, Higgs & off-shell quantities

- Easy to include fermions in 4d, e.g. gluon-quark amplitudes in massless QCD

\[ M_{n,k}(g; q\bar{q}) = \int d\mu_{n,k} \, PT_n \, J_{\text{ferm}}, \text{ Jac. depends on helicities & flavors of quarks only!} \]

follow from gluon-gluino ones in SYM → new rep for all QCD tree amplitudes

- Higgs mechanism in CHY? very different from massive CHY from KK reduction

1\textsuperscript{st} step: Higgs + n gluons as (n+2) on-shell legs \[ p_\phi = \lambda\tilde{\lambda} + \mu\tilde{\mu} \]

\[ M_{n+1,k}(\phi; n_g) = m_H^4 \int d\mu'_{n+2,k} \, PT_n, \quad \sigma_\lambda, \sigma_\mu \text{ fixed}; \lambda, \mu \text{ eqs removed.} \]

- Opens up CHY for SM amps and off-shell quantities (form factors, correlators...)

Jac. depends on helicities & flavors of quarks only!
Summary & outlook

- **New picture**: gluons (massless particles) scattering via punctures on a sphere. Suggest a weak-weak duality of QFT and string theory for S-matrix?

- **Complimentary to FD’s**: \((n-3)!\) virtual amplitudes with all symmetries manifest!

- **Web of theories** connected by operations e.g. \(\oplus\) (interaction) & \(\otimes\) (double-copy)

- **Loops**: higher-genus vs. higher-punctures; integrands vs. integrated amps

- Massive theories, off-shell quantities etc. **Scope of QFT’s** with natural CHY formula?

- **S-matrix** as **representation theory**, of \((\text{Poincare} \subset \text{BMS} \subset)\) some group?
Thank You!