Quantum Integrable Systems from Conformal Blocks
(arXiv: 1605.05105 with Joshua D. Qualls)

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Strings 2016 @ Beijing
This talk attempts to discuss the following questions:

▸ Are there more systematic or even more efficient ways to obtain conformal blocks in various dimensions and set up?

▸ Can one extend the correspondence between the degenerate correlation functions and quantum integrable systems in $d = 2$ dim. CFTs to $d > 2$ dim. CFTs?

▸ If so, how general such a correspondence is?
Let us begin with the four point function of scalar conformal primary operator $\phi_i(x)$ with scale dimension $\Delta_i$ in $d$-dim. CFTs:

$$
< \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) > = \left( \frac{x_{14}^2}{x_{24}^2} \right)^a \left( \frac{x_{14}^2}{x_{13}^2} \right)^b \frac{F(u,v)}{(x_{12}^2)^{(\Delta_1+\Delta_2)/2} (x_{34}^2)^{(\Delta_3+\Delta_4)/2}}
$$

where $x_{ij} = x_i - x_j$ $a = \frac{\Delta_2-\Delta_1}{2}$, $b = \frac{\Delta_3-\Delta_4}{2}$ and $F(u,v)$ is a function of conformally invariant cross ratios:

$$
u = \frac{x_{12}x_{34}}{x_{13}^2x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}x_{23}}{x_{13}^2x_{24}^2} = (1-z)(1-\bar{z}).
$$

Decomposing the four point function further into contributions from the individual exchanged primary operators $O_{\Delta,l}$ and its descendants:

$$
< \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) > = \sum_{\{O_{\Delta,l}\}} \lambda_{12} O_{\Delta,l} \lambda_{34} O_{\Delta,l} \mathcal{W}_{O_{\Delta,l}}(x_i)
$$

$\mathcal{W}_{O_{\Delta,l}}(x_i)$ is “conformal partial wave” and $\lambda_{ij} O_{\Delta,l}$ are “OPE coefficients.”.
Conformal invariance also fixes $\mathcal{W}_{\Delta, l}(x_i)$ into:

$$\mathcal{W}_{\mathcal{O}_{\Delta, l}}(x_i) = \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{G_{\mathcal{O}_{\Delta, l}}(u, v)}{(x_{12}^2)^{\frac{(\Delta_1 + \Delta_2)}{2}} (x_{34}^2)^{\frac{(\Delta_3 + \Delta_4)}{2}}}$$

where $G_{\mathcal{O}_{\Delta, l}}(u, v)$ is the “Conformal Block” for $\mathcal{O}_{\Delta, l}$ conformal family.

The four point functions need to satisfy crossing symmetry equation when e. g. $\phi_1(x_1) \leftrightarrow \phi_3(x_3)$:

$$\sum_{\{\mathcal{O}_{\Delta, l}\}} \lambda_{12\mathcal{O}_{\Delta, l}} \lambda_{34\mathcal{O}_{\Delta, l}} G_{\mathcal{O}_{\Delta, l}}(u, v) = u^{\frac{\Delta_1 + \Delta_2}{2}} v^{\frac{\Delta_2 + \Delta_3}{2}} \sum_{\{\mathcal{O}'_{\Delta, l}\}} \lambda_{14\mathcal{O}'_{\Delta, l}} \lambda_{32\mathcal{O}'_{\Delta, l}} G_{\mathcal{O}'_{\Delta, l}}(v, u)$$

For unitary CFTs, if we know $G_{\mathcal{O}_{\Delta, l}}(u, v)$ exactly or at least some approximate forms, then assuming $\lambda_{12\mathcal{O}_{\Delta, l}} \lambda_{34\mathcal{O}_{\Delta, l}} \geq 0$, and start numerically putting bounds on spectrum of $\{\Delta_{\mathcal{O}}\}$.

[See Rychkov, Simmons-Duffin 16 for reviews]
Determining $G_{\Delta, l}(u, v)$ for general four point functions remains difficult, one way is to consider “quadratic Casimir operator”: \[ \hat{C}_2 = \frac{1}{2} L^{AB} L_{AB} = \frac{1}{2} (L_1 + L_2)_{AB} (L_1 + L_2)^{AB} \] where $L_{i, AB}$ is Lorentz generator in $d + 2$-dimensional embedding space.

For scalar primaries, we can define following differential operators:

\[ D_z^{(a, b, c)} = z^2 (1 - z) \partial_z^2 - ((a + b + 1)z^2 - cz) \partial_z - abz, \]
\[ D_{\bar{z}}^{(a, b, c)} = \bar{z}^2 (1 - \bar{z}) \partial_{\bar{z}}^2 - ((a + b + 1)\bar{z}^2 - c\bar{z}) \partial_{\bar{z}} - ab\bar{z}, \]

\[ \Delta_2^{(\varepsilon)}(a, b, c) = D_z^{(a, b, c)} + D_{\bar{z}}^{(a, b, c)} + 2\varepsilon \frac{z\bar{z}}{z - \bar{z}} ((1 - z) \partial_z - (1 - \bar{z}) \partial_{\bar{z}}), \]

where $\varepsilon = \frac{d-2}{2}$ enters as free parameter.

Setting $G_{\Delta, l}(u, v) = F_{\lambda_+ \lambda_-}^{(\varepsilon)}(z, \bar{z})$ which is symmetric $z \leftrightarrow \bar{z}$, the action of $\hat{C}_2$ is

\[ \Delta_2^{(\varepsilon)}(a, b, 0) \cdot F_{\lambda_+ \lambda_-}^{(\varepsilon)}(z, \bar{z}) = c_2(\lambda_+, \lambda_-) F_{\lambda_+ \lambda_-}^{(\varepsilon)}(z, \bar{z}), \quad \lambda_{\pm} = \frac{\Delta \pm l}{2}. \]
In addition, we also have “quartic Casimir operator”:

\[ \hat{C}_4 = \frac{1}{2} L^{AB} L_{BC} L^{CD} L_{DA}. \]  

(10)

The action of \( \hat{C}_4 \) on primary scalars is also expressed as eigen-equation:

\[ \Delta^{(e)}_{4}(a, b, 0) \cdot F^{(e)}_{\lambda_+ \lambda_-}(z, \bar{z}) = c_4(\lambda_+, \lambda_-) F^{(e)}_{\lambda_+ \lambda_-}(z, \bar{z}), \]  

(11)

\[ \Delta^{(e)}_{4}(a, b, c) = \left[ \frac{z\bar{z}}{z - \bar{z}} \right]^{2\epsilon} \left[ D^{(a,b,c)}_{z} - D^{(a,b,c)}_{\bar{z}} \right] \left[ \frac{z - \bar{z}}{z\bar{z}} \right]^{2\epsilon} \left[ D^{(a,b,c)}_{z} - D^{(a,b,c)}_{\bar{z}} \right]. \]  

(12)

The quadratic and quartic Casimir operators are by definition commuting:

\[ [\hat{C}_2, \hat{C}_4] = 0 \implies [\Delta^{(e)}_{2}(a, b, c), \Delta^{(e)}_{4}(a, b, c)] = 0. \]  

(13)
Explicit closed expressions of scalar conformal blocks are only available in even-dim. CFTs in terms of following functions [Dolan-Osborn 11]:

\[
F_{\lambda_+\lambda_-}^{\pm}(z,\bar{z}) = g_{\lambda_+}(z)g_{\lambda_-}(\bar{z}) \pm g_{\lambda_+}(\bar{z})g_{\lambda_-}(z),
\]

(14)

\[
g_{\lambda}(x) = z^{\lambda/2}F_1(a + \lambda, b + \lambda; 2\lambda; x).
\]

(15)

\[
d = 2/\varepsilon = 0 \quad : \quad F_{\lambda_+\lambda_-}^{(0)}(z,\bar{z}) = \frac{1}{2}F_{\lambda_+\lambda_-}^{+}(z,\bar{z}),
\]

\[
d = 4/\varepsilon = 1 \quad : \quad F_{\lambda_+\lambda_-}^{(1)}(z,\bar{z}) = \frac{1}{l+1} \left( \frac{z\bar{z}}{z-\bar{z}} \right) F_{\lambda_+\lambda_-}^{-}(z,\bar{z}),
\]

\[
d = 6/\varepsilon = 2 \quad : \quad F_{\lambda_+\lambda_-}^{(2)}(z,\bar{z}) = 5 \text{ terms of } F_{\lambda_+\lambda_-}^{-}(z,\bar{z}).
\]

For general \(\varepsilon\) however, relying on iterative approach/recurrence relation [Rychkov-Hogervorst 13, Costa-Hansen-Penedones-Trevisani 16]

\[
G_{\Delta,l}(r, \cos \theta) = \sum_{n=0}^{\infty} \sum_{j} B_{n,j} r^{\Delta+n} \hat{C}_j^\varepsilon(\cos \theta), \quad B_{n,j} \geq 0
\]

(16)

where \(re^{i\theta} = \frac{z}{(1+\sqrt{1-z})^2}\) and \(\hat{C}_j^\varepsilon(x)\) is normalized Gegenbauer polynomial.
Here we consider a quantum integrable system given by Hamiltonian:

\[
\hat{H}_{BC_2} = - \left( \frac{\partial^2}{\partial u^2} + \frac{\partial}{\partial \bar{u}^2} \right) + 2a \left( \frac{(a - K_{u\bar{u}})}{\sinh^2(u - \bar{u})} + \frac{(a - \tilde{K}_{u\bar{u}})}{\sinh^2(u + \bar{u})} \right) \\
+ \left( \frac{b(b - K_u)}{\sinh^2 u} - \frac{b'(b' - K_u)}{\cosh^2 u} \right) + \left( \frac{b(b - K_{\bar{u}})}{\sinh^2 \bar{u}} - \frac{b'(b' - K_{\bar{u}})}{\cosh^2 \bar{u}} \right).
\]

This is called “Hyperbolic Calogero-Sutherland spin chain of BC$_2$, where

Permutation : $K_{u\bar{u}} f(u, \bar{u}) = f(\bar{u}, u)$, $\tilde{K}_{u\bar{u}} f(u, \bar{u}) = f(-\bar{u}, -u)$,

Reflection : $K_u f(u, \bar{u}) = f(-u, \bar{u})$, $K_{\bar{u}} f(u, \bar{u}) = f(u, -\bar{u})$. 
The quantum integrability of a Quantum Integrable System is ensured by the existence of commuting conserved charges:

\[
[\hat{I}_j, \hat{I}_k] = 0, \quad \frac{d\hat{I}_k}{dt} = [\hat{I}_k, \hat{H}] = 0, \quad k = 1, \ldots, N. \tag{17}
\]

They are most easily constructed from the commuting Dunkl operators:

[Finkel et al 12]

\[
\hat{J}_u^{(a)} = \frac{\partial}{\partial u} - \left[ \frac{b}{\tanh \frac{u}{2}} + \frac{b'}{\coth \frac{u}{2}} \right] K_u - a \left[ \frac{\tilde{K}_{u\bar{u}}}{\tanh \frac{u+\bar{u}}{2}} + \frac{K_{u\bar{u}}}{\tanh \frac{u-\bar{u}}{2}} \right],
\]

\[
\hat{J}_{\bar{u}}^{(a)} = \frac{\partial}{\partial \bar{u}} - \left[ \frac{b}{\tanh \frac{\bar{u}}{2}} + \frac{b'}{\coth \frac{\bar{u}}{2}} \right] K_{\bar{u}} - a \left[ \frac{\tilde{K}_{u\bar{u}}}{\tanh \frac{u+\bar{u}}{2}} - \frac{K_{u\bar{u}}}{\tanh \frac{u-\bar{u}}{2}} \right],
\]

There are two independent commuting integrals of motion:

\[
\hat{\mathcal{L}}_2 = \hat{H}_{BC2} = - \left( \hat{J}_u^{(a)} \right)^2 - \left( \hat{J}_{\bar{u}}^{(a)} \right)^2, \quad \hat{\mathcal{L}}_4 = - \left( \hat{J}_u^{(a)} \right)^4 - \left( \hat{J}_{\bar{u}}^{(a)} \right)^4. \tag{18}
\]
Now we establish the following exact mapping between QIS and CFT:
[Isachenkov-Schomerus 16, HYC-Qualls 16]

\[
\begin{align*}
\left[ \Delta_2^{(\varepsilon)}(a, b, c), \Delta_4^{(\varepsilon)}(a, b, c) \right] &= 0 \quad \iff \quad \left[ \hat{I}_2^{(BC_2)}, \hat{I}_4^{(BC_2)} \right] = 0, \\
G_{\Delta, l}(z, \bar{z}) &\iff \psi_{\lambda^+ \lambda^-}^{(\varepsilon)}(u, \bar{u}).
\end{align*}
\]

The first step is to look for the appropriate commutator-preserving similarity transformation:

\[
\begin{align*}
\Delta_{2,4}^{(\varepsilon)}(a, b, c) &\longrightarrow \chi_{a,b,c}^{(\varepsilon)}(z, \bar{z}) \Delta_{2,4}^{(\varepsilon)}(a, b, c) \frac{1}{\chi_{a,b,c}^{(\varepsilon)}(z, \bar{z})} \\
&= \chi_{a,b,c}^{(\varepsilon)}(z, \bar{z}) \Delta_{2,4}^{(\varepsilon)}(a, b, c) \frac{1}{\chi_{a,b,c}^{(\varepsilon)}(z, \bar{z})} \\
&= (19)
\end{align*}
\]

This allows us to relate CFT Casimirs to the commuting quantum integrals of motion. The desired transformation is given by the following double-cover map:

\[
\begin{align*}
z(u) &= -\frac{1}{\sinh^2 u} \quad \iff \quad e^{u(z)} = -\frac{z}{(1 + \sqrt{1 - z})^2}, \quad \text{c.f. radial expansion} \\
\chi_{a,b,c}^{(\varepsilon)}(z(u), \bar{z}(\bar{u})) &= \frac{[\frac{1}{z(u)}(1 - z(u))]}{[z(u)\bar{z}(\bar{u})]^{\frac{1-c}{2}}}[z(u) - \bar{z}(\bar{u})]^{\varepsilon}.
\end{align*}
\]
Direct computation shows when acting on symmetric $f(u, \bar{u}) = f(\bar{u}, u)$:

\[
\chi_{a,b,c}^{(\varepsilon)}(z, \bar{z}) \Delta_2^{(\varepsilon)}(a, b, c) \frac{1}{\chi_{a,b,c}^{(\varepsilon)}(z, \bar{z})} = -\frac{1}{4} \mathcal{I}_2 = -\frac{1}{4} H_{BC2},
\]

\[
a = \varepsilon, \quad b = (a - b) + \frac{1}{2}, \quad b' = (a + b - c) + \frac{1}{2}. \tag{20}
\]

Notice when $\varepsilon = 0$ or $\varepsilon = 1$, i.e. $d = 2$ or $d = 4$, pair-wise interactions both vanish (Pöschl-Teller). More generally we have the correspondence:

\[
\psi_{\lambda, \lambda}^{(\varepsilon)}(u, \bar{u}) = \chi_{a,b,c}^{(\varepsilon)}(z(u), \bar{z}(\bar{u})) F_{\lambda, \lambda}^{(\varepsilon)}(z(u), \bar{z}(\bar{u}))
\]

The eigenfunction is manifestly symmetric under $u \leftrightarrow \bar{u}$, and has been constructed by Koornwinder and collaborators, and we obtain:

\[
\frac{(z\bar{z})^a}{(-4)^a \Delta(16)^a} F_{\lambda, \lambda}^{(\varepsilon)}(z, \bar{z}) = e^{-(\chi + a)(u + \bar{u})} \times \lim_{q \to 1^{-}} \hat{K}_{L, \bar{L}}^{(2)}(e^u, e^{\bar{u}}; q, q^\varepsilon, q^{-\chi-a}, q^{a-b-1}, -q^{a+b+1}, 1, -1).
\]

$(L, \bar{L})$ are two row partitions, $L - \bar{L} = l$, $\bar{L} = \frac{1}{2}[(\Delta - l)]$, $\chi = \frac{1}{2}(\Delta - l) - \bar{L}$. 

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The quartic Casimir $\Delta_4^{(\varepsilon)}(a, b, c)$ is also expressed in terms of $\hat{I}_2$ and $\hat{I}_4$. First we can show that acting on $\psi_{\lambda^+\lambda^-}(u, \bar{u})$:

$$
\left( J_u^{(0)} \right)^2 - \left( J_{\bar{u}}^{(0)} \right)^2 = 4 \chi_{a,b,c}^{(0)}(z, \bar{z}) \left[ D_z^{(a,b,c)} - D_{\bar{z}}^{(a,b,c)} \right] \frac{1}{\chi_{a,b,c}^{(0)}(z, \bar{z})}. \quad (21)
$$

Next define following combinations of Dunkl and permutation operators:

$$
\hat{L}_{uu\bar{u}}^{(+)} = \hat{J}_u^{(\varepsilon)} + \hat{J}_{\bar{u}}^{(\varepsilon)} + 2\varepsilon \hat{K}_{u\bar{u}}, \quad \hat{L}_{uu\bar{u}}^{(-)} = \hat{J}_u^{(\varepsilon)} - \hat{J}_{\bar{u}}^{(\varepsilon)} + 2\varepsilon \hat{K}_{u\bar{u}}
$$

We can show that:

$$
\hat{L}_{uu\bar{u}}^{(+)} \hat{L}_{uu\bar{u}}^{(-)} \psi_{\lambda^+\lambda^-}^{(\varepsilon)}(u, \bar{u}) = t^{(\varepsilon)}(z, \bar{z}) \left[ \left( J_u^{(0)} \right)^2 - \left( J_{\bar{u}}^{(0)} \right)^2 \right] \frac{1}{t^{(\varepsilon)}(z, \bar{z})} \psi_{\lambda^+\lambda^-}^{(\varepsilon)}(u, \bar{u}),
$$

$$
t^{(\varepsilon)}(z(u), \bar{z}(\bar{u})) = \left[ \sinh(u + \bar{u}) \sinh(u - \bar{u}) \right]^{\varepsilon}. \quad (22)
$$

Using the invariance of $\psi_{\lambda^+\lambda^-}^{(\varepsilon)}(u, \bar{u})$ under reflection and permutation.
While above expression is invariant under $K_u$ and $K_{\bar{u}}$, however crucially:

$$K_{u\bar{u}} \hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)} (u, \bar{u}) = -\hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)} (u, \bar{u}),$$

$$\tilde{K}_{u\bar{u}} \hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)} (u, \bar{u}) = -\hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)} (u, \bar{u}),$$

These properties in turn imply:

$$\hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)}$$

$$= \frac{1}{t^{(\epsilon)}(z, \bar{z})} \left[ \left( J^{(0)}_u \right)^2 - \left( J^{(0)}_{\bar{u}} \right)^2 \right] t^{(\epsilon)}(z, \bar{z}) \hat{L}^{(+)}_{u\bar{u}} \hat{L}^{(-)}_{u\bar{u}} \psi_{\lambda+\lambda_-}^{(\epsilon)}$$

$$= \frac{1}{t^{(\epsilon)}(z, \bar{z})} \left[ \left( \hat{J}^{(0)}_u \right)^2 - \left( \hat{J}^{(0)}_{\bar{u}} \right)^2 \right] t^{(\epsilon)}(z, \bar{z})^2 \left[ \left( \hat{J}^{(0)}_u \right)^2 - \left( \hat{J}^{(0)}_{\bar{u}} \right)^2 \right] \frac{\psi_{\lambda+\lambda_-}^{(\epsilon)}}{t^{(\epsilon)}(z, \bar{z})}.$$  

This precisely equals to the transformed $\Delta_4^{(\epsilon)}(a, b, c)$ and expanding

$$\chi_{a, b, c}^{(\epsilon)}(z, \bar{z}) \Delta_4^{(\epsilon)}(a, b, c) \frac{1}{\chi_{a, b, c}^{(\epsilon)}(z, \bar{z})} = -\frac{1}{8} \mathcal{I}_4 + \frac{1}{16} \mathcal{I}_2^2 + \frac{\epsilon^2}{2} \mathcal{I}_2 + \epsilon^4. \quad (23)$$
Generalizations

We can extend the analysis to scalar conformal blocks in SCFTs using their Casimir operators, e. g. for four SUSY [Bobev et al 15]:

$$\Delta_2^{\varepsilon}(a+1, b, 1) \cdot F_{\lambda_+\lambda_-}^{4\text{susy}}(z, \bar{z}) = c_{2}^{4\text{susy}}(\lambda_+ , \lambda_-) F_{\lambda_+\lambda_-}^{4\text{susy}}(z, \bar{z})$$  (24)

Identical hyperbolic CS system arises after the similarity transformation. Similarly for eight SUSYs, the eigenfunctions are related to non-SUSY ones via a multiplicative factor and simple parameter shifts.

Viewing scalar conformal blocks as the orthogonal eigenfunctions of hyperbolic CS system, they serve as natural basis for expanding other conformal blocks once space-time spins are taken care of.

$$\psi_{e}^{(p)}(u, \bar{u}) = \frac{1}{[\sinh(u + \bar{u}) \sinh(u - \bar{u})]^{2p}} \sum_{(m,n) \in Oct_{e}^{(p)}} c_{m,n}^{e} \psi_{\rho_1 + m, \rho_2 + n}^{a_{e}, b_{e}, c_{e}}(u, \bar{u}),$$

[Echeverri et al 15, 16]
Future Directions

- Do higher point correlation functions also have underlying correspondence with quantum integrable systems.
- Can the correspondence with QIS extends to Virasoro or $W_N$ blocks in 2d CFTs beyond degenerate vertex operators?
- Is this connection with quantum integrable system accidental? Via AdS/CFT conformal block computations, can we see similar structures in gravity side? 2 points $\rightarrow$ 3 points $\rightarrow$ 4 points?