Frog Model with Drift on $\mathbb{T}_d$

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The Frog Model

- Let G be a connected graph with root $\varnothing$.
- On each vertex $v \neq \varnothing$, there are η_v sleeping ("inactive") frogs.
- At $t = 0$, frogs at $\varnothing$ wake up ("active") and perform indep. RWs.
- When an active frog hits a vertex with sleeping frogs, it wakes up all sleeping frogs on that vertex.
- Frogs perform independent RWs upon waking up.

$G = \mathbb{Z}^2$, Poi(1) frogs, SRW

Recurrence & Transience

- A "zero-one-law":

$$\mathbb{P}(\{\varnothing \text{ is visited infinitely often}\}) = \underbrace{0} \text{ or } \underbrace{1}.$$

[Kosygina & Zerner, 17] for $\mathbb{Z}^d$;
[Hoffman, Johnson, & Junge, 17] for $\mathbb{T}_d$.

- When the frog model is recurrent? It depends on
  - # of frogs on each site (distribution of η_v);
  - type of the random walk (biased? unbiased?);
  - degree (distribution) of the underlying random graph;

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Frog model on $\mathbb{Z}^d$

- [Telcs & Wormald, 99] On $\mathbb{Z}^d$ with SRW and $\eta_v \equiv 1$ for all v, the frog model is **recurrent** for any d.
- [Gantert & Schmidt, 09] On $\mathbb{Z}$ with i.i.d. η_v frogs per site, if RW has a drift to the right,
$$\mathbb{E}\left[(\log \eta_v)_+\right] = \infty \quad \Longleftrightarrow \quad \text{recurrent.}$$
- [Döbler & Pfeifroth, 14] On $\mathbb{Z}^d$ with $d \geq 2$,
$$\mathbb{E}\left[(\log \eta_v)_+^{\frac{d+1}{2}}\right] = \infty \quad \Longrightarrow \quad \text{recurrent.}$$
- [Döbler, Gantert, Höfelsauer, Popov & Weidner, 17] On $\mathbb{Z}^d$ with $\eta_v \equiv 1$: $\exists$ a phase transition depending on (a) the total probability on the $\pm e_1$ direction and (b) the drift α in the e_1 direction of the RW. Different behavior for $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

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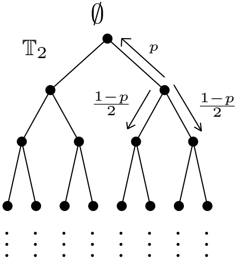
Frog model on trees

- [Hoffman, Johnson, & Junge, 17]:
  $\mathrm{FM}_d :=$ frog model on $\mathbb{T}_d$ with $\eta_v \equiv 1$ and SRW
  - recurrent if $d = 2$;
  - transient if $d \geq 5$;
  - unclear for $d = 3, 4$.
- [Hoffman, Johnson, & Junge, 16]:
  frog model on $\mathbb{T}_d$ with $\eta_v \overset{\text{i.i.d.}}{\sim} \mathrm{Pois}(\mu)$ and SRW: $\exists\ \mu_d$ s.t.
  - recurrent if $\mu > \mu_d$;
  - transient if $\mu < \mu_d$;
  - and $Cd < \mu_d < C'd \log d$.
- [Rosenberg, 18]: recurrent on (2,3)-alternating tree with $\eta_v \equiv 1$.
- [Michelen & Rosenberg, 19]:
  for $\eta_v \overset{\text{i.i.d.}}{\sim} \mathrm{Pois}(\mu)$ on GW trees: phase transtion.

$FM(d,p)$: frog model on $\mathbb{T}_d$ with drift p

- $\mathrm{FM}(d,p)$: $\eta_v \equiv 1$ on $\mathbb{T}_d$.
  Each active frog moves toward the root with prob. p, and otherwise to a random child vertex.



- [Hoffman, Johnson, & Junge, 17]
  $\mathrm{FM}_d := \mathrm{FM}(d, \frac{1}{d+1})$
  $d = 2$: recurrent; $p_2 = \frac{1}{3}$

- $p_d := \inf\{p : \mathrm{FM}(d,p) \text{ is recurrent}\} = \inf\{p : V_{d,p} = \infty \text{ a.s.}\}$.
  $V_{d,p} := \#$ frog visits to the root vertex.

- Obviously, $\frac{1}{d+1} \le p_d \le \frac{1}{2}$. **Better bounds?**

Conjecture 1 (monotonicity). For all $d \ge 2$, $p_{d+1} \le p_d$.

If the Conjecture is true, the best universal upper bound $p_d \le 1/3$.

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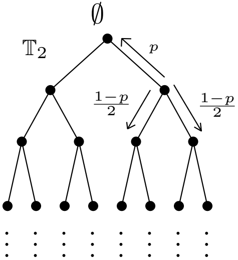

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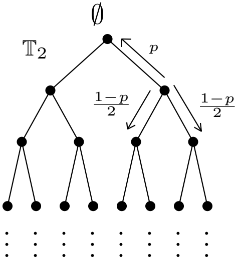

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Theorem (Guo, T.,& Wei, 20)

$p_d \leq 1/3$ for all $d \geq 2$.

General Strategy:

- Construct a process $\mathcal{P}_1 \preceq \mathrm{FM}(d,p)$. $\mathcal{P}_1$ is recurrent $\Rightarrow p_d \leq p$.
- Construct a process $\mathcal{P}_2 \succeq \mathrm{FM}(d,p)$. $\mathcal{P}_2$ is transient $\Rightarrow p \leq p_d$. $(*)$

Direct coupling of two frog models seems hard:

Conjecture 2. $\mathrm{FM}(d,p) \preceq \mathrm{FM}(d+1,p)$.
Remark. This would imply Conjecture 1.
However, critical parameter is not always a monotonic function of the graph, e.g., [Fontes, Machado, Sarkar, 04]

Conjecture 3. $\mathrm{FM}(d,p') \preceq \mathrm{FM}(d,p)$ if $p' \leq p$.
Remark. This would imply "only one phase transition".

Improved lower bound: coupling with BRW

Proposition. $p_d \geq \frac{2-\sqrt{2}}{4} \approx 0.1464$.

Sketch. Define $N_t^{FM} := \#\{\text{active frogs at time } t\}$ in $FM(d,p)$ and $N_t^{BRW} := \#\{\text{particles at time } t \text{ in a BRW on } \mathbb{Z}_+\}$, where particles do not split when moving to the left (with prob. p) but always split in two when moving to the right. This BRW is equivalent to FM(∞, p).

Then,

$$N_t^{FM} \preceq N_t^{BRW}.$$

This BRW is transient if $p < \frac{2-\sqrt{2}}{4}$ and so is $FM(d,p)$.
Hence $p_d \geq \frac{2-\sqrt{2}}{4}$ for all d.

Conjecture 4. $\lim_{d\to\infty} p_d = \frac{2-\sqrt{2}}{4}$. ("no gap")

Upper bound:

Idea: for each p, construct a self-similar frog model $\mathrm{SFM}(d, \rho_p)$ so that

$$\mathrm{SFM}(d, \rho_p) \preceq \mathrm{FM}(d, p).$$

Show that $\mathrm{SFM}(d, \rho_{1/3})$ is recurrent, and then $p_d \leq 1/3$.

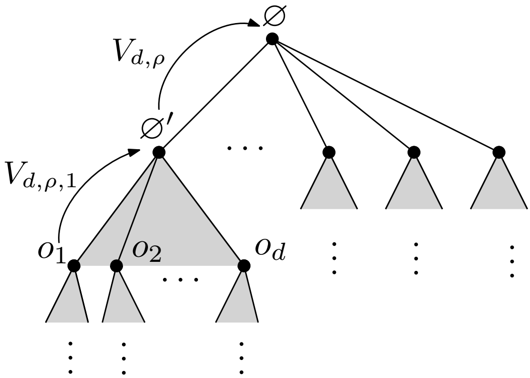
Self-similar Frog Model:

(i) Active frogs move toward the root with prob. ρ or otherwise away from the root to a random child vertex.

(ii) Frog can not visit the vertex that it comes from in the next step. ("nonbacktracking")

(iii) Active frogs are removed upon visiting the root.

(iv) For each subtree attached to the sub-root vertex $\emptyset$, only one active frog is allowed to enter there.



Self-similarity: $(V_{d,\rho,i} | o_i$ is ever visited$)$ are i.i.d. copies of $V_{d,\rho}$.

$$V_{d,\rho} \stackrel{\mathcal{D}}{=\!=} \sum_{i=1}^{d} \mathrm{Bin}\left(V_{d,\rho,i}, \frac{\rho}{\rho + \frac{d-1}{d}(1-\rho)} \right) + \mathbb{1}\{f_{\varnothing'} \to \varnothing\}.$$

Coupling: FM(d, p) and SFM(d, ρ_p)

FM$(d, p) \longrightarrow$ remove all loops in frogs' random walk paths

$\longrightarrow$ imposing rules (iii) and (iv)

$\longrightarrow$ SFM(d, ρ_p) with $\frac{\rho_p}{\rho_p + \frac{d-1}{d}(1-\rho_p)} = \frac{p}{1-p}$

$$\therefore \quad \text{FM}(d, p) \succeq \text{SFM}(d, \rho_p).$$

Remark. If $p = \frac{1}{d+1}$, then $\rho_{1/(d+1)} = \frac{1}{d+1} (= p)$.
And for $p = 1/3$ then $\rho_{1/3} = \frac{d-1}{2d-1}$.

Proof of $p_d \leq \frac{1}{3}$:

- It is sufficient to prove $\mathrm{SFM}(d, \frac{d-1}{2d-1})$ is recurrent.
- Let V_d be the number of visits to the root in $\mathrm{SFM}(d, \frac{d-1}{2d-1})$ and $g_d(x) = \mathbb{E}(x^{V_d})$ be the generating function.
  Then $g_d(x) \equiv 0$ on $[0, 1)$ implies $V_d = \infty$ a.s..
- Find an operator $\mathcal{A}_d$ such that $\mathcal{A}_d g_d = g_d$.
  "Recursive Distributional Equation" (RDE), due to "self-similarity".
- Prove good properties of $\mathcal{A}_d$.
- This is a classical method, used to prove, e.g.,
  [Hoffman, Johnson, & Junge, 17]: $\mathrm{SFM}(2, 1/3)$ is recurrent
  [Rosenberg, 18]: FM is recurrent on (2,3)-alternating tree.

Good Properties of $\mathcal{A}_d$:

(i) Recall: proof of "SFM$(2, 1/3)$ is recurrent".

- $g_2 = \mathcal{A}_2 g_2$ ("fixed point").
- $\mathcal{F} := \{f : [0,1] \to [0,1],\ \text{nondecreasing}\}$. $\mathcal{A}_2$ maps $\mathcal{F}$ to $\mathcal{F}$
- $\mathcal{A}_2$ is monotone: if $f \le g$, then $\mathcal{A}_2 f \le \mathcal{A}_2 g$.
- $\mathcal{A}_2^n 1 \to 0$ as $n \to \infty$.

$\Longrightarrow \quad g_2 = \mathcal{A}_2^n g_2 \le \mathcal{A}_2^n 1 \to 0.$

(ii) We would only need

$$\mathcal{A}_d g_d \le \mathcal{A}_2 g_d$$

because this would imply

$$g_d = \mathcal{A}_d g_d \le \mathcal{A}_2 g_d = \mathcal{A}_2 \mathcal{A}_d g_d \le \mathcal{A}_2^2 g_d \le \cdots \le \mathcal{A}_2^n g_d \le \mathcal{A}_2^n 1 \to 0.$$

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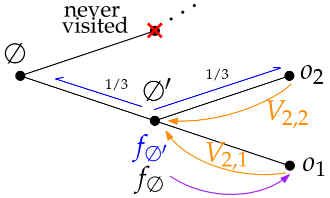
Base case: $d = 2$



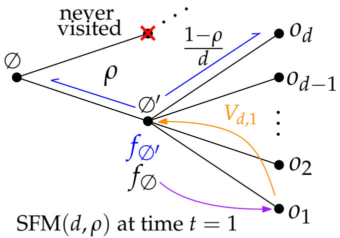
SFM$(2, 1/3)$ at time $t = 1$

$$D := \{f_{\varnothing'} \text{ activates a new branch}\}$$
$$E_k := \{k \text{ children of } \varnothing' \text{ are activated}\}$$
$$A_k := D \cap E_k; \quad B_k := D^c \cap E_k$$

$$g_2(x) = \mathbb{E}(x^{V_2}) = \mathbb{E}(x^{V_2}\mathbb{1}_{A_2}) + \mathbb{E}(x^{V_2}\mathbb{1}_{B_1}) + \mathbb{E}(x^{V_2}\mathbb{1}_{B_2})$$
$$= \underbrace{\tfrac{1}{3}g_2^2\left(\tfrac{x+1}{2}\right) + \tfrac{x+1}{3}g_2\left(\tfrac{x}{2}\right) + \tfrac{x+1}{3}g_2\left(\tfrac{x+1}{2}\right)\left[g_2\left(\tfrac{x+1}{2}\right) - g_2\left(\tfrac{x}{2}\right)\right]}_{\mathcal{A}_2 g_2(x)},$$

using $\mathbb{E}(x^{\mathrm{Bin}(V_2, \frac{1}{2})}) = g_2(\frac{x+1}{2})$, $\mathbb{E}(x^{\mathbb{1}\{f_{\varnothing'} \to \varnothing\}}\mathbb{1}_{D^c}) = \frac{x+1}{3}$, and $\mathbb{E}(x^{\mathrm{Bin}(V_{2,1}, \frac{1}{2})}\mathbb{1}\{\text{no frogs in } \mathbb{T}_2(o_1) \text{ enter } \mathbb{T}_2(o_2)\}) = g_2(\frac{x}{2})$.

Finding $\mathcal{A}_d$:



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Key observation 1: for $L \subseteq \{o_2, \ldots, o_d\}$, $|L| = \ell \in \{0, 1, \ldots, d-1\}$:

$$\mathbb{E}[x^{V_{d,1}} \mathbb{1}\{\text{no frogs in } \mathbb{T}_d(o_1) \text{ enter } \mathbb{T}_d(o_i), o_i \in L\}] = g_d \underbrace{\left(\frac{x}{2} + \frac{d-\ell-1}{2(d-1)}\right)}_{c_d^{(d-\ell-1)}(x)}$$

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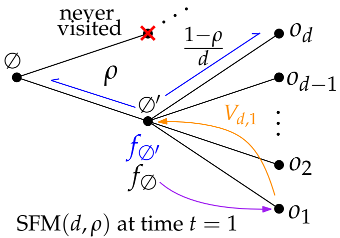

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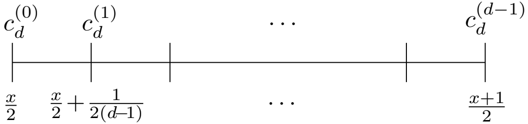

Key observation 2: $\mathbb{E}(x^{V_d} 1_{A_k})$ and $\mathbb{E}(x^{V_d} 1_{B_k})$ are polynomials in k variables $g_d \circ c_d^{(0)}(x), \ \ldots, \ g_d \circ c_d^{(k-1)}(x)$ of degree k.

Key observation 3: the recursion.

$$g_d(x) = \mathbb{E}(x^{V_d}) = \sum_{k=2}^{d} \mathbb{E}(x^{V_d} \mathbf{1}_{A_k^{(d)}}) + \sum_{k=1}^{d} \mathbb{E}(x^{V_d} \mathbf{1}_{B_k^{(d)}})$$

same "structure" $\times$ combinatorial factor

$$g_{d+1}(x) = \mathbb{E}(x^{V_{d+1}}) = \sum_{k=2}^{d} \mathbb{E}(x^{V_{d+1}} \mathbf{1}_{A_k^{(d+1)}}) + \sum_{k=1}^{d} \mathbb{E}(x^{V_{d+1}} \mathbf{1}_{B_k^{(d+1)}})$$

$$+ \underbrace{\mathbb{E}(x^{V_{d+1}} \mathbf{1}_{A_{d+1}^{(d+1)}})}_{\text{function of } \mathbb{E}(x^{V} \mathbf{1}_{A_k^{(d+1)}}),\ k=1,2,\ldots,d} + \underbrace{\mathbb{E}(x^{V_{d+1}} \mathbf{1}_{B_{d+1}^{(d+1)}})}_{\text{function of } \mathbb{E}(x^{V} \mathbf{1}_{B_k^{(d+1)}}),\ k=1,2,\ldots,d}$$

The recursive algorithm: an example

$$\begin{aligned} g_2(x) &= \tfrac{1}{3} g_2^2\left(\tfrac{x+1}{2}\right) + \tfrac{x+1}{3} g_2\left(\tfrac{x}{2}\right) + \tfrac{x+1}{3} g_2\left(\tfrac{x+1}{2}\right)\left[g_2\left(\tfrac{x+1}{2}\right) - g_2\left(\tfrac{x}{2}\right)\right] \\ &= \tfrac{1}{3} P_2(z_1, z_2) + \tfrac{x+1}{3}\left[Q_1(z_1) + Q_2(z_1, z_2)\right] \end{aligned}$$

where $P_2(z_1, z_2) = z_2^2$, $Q_1(z_1) = z_1$, and $Q_2(z_1, z_2) = z_2(z_2 - z_1)$
with $z_1 = g_2 \circ c_2^{(0)}(x) = \frac{x}{2}$ and $z_2 = g_2 \circ c_2^{(1)}(x) = \frac{x+1}{2}$.

$$\begin{aligned} g_3(x) &= \mathbb{E}(x^{V_3} \mathbb{1}_{A_2}) + \mathbb{E}(x^{V_3} \mathbb{1}_{B_1}) + \mathbb{E}(x^{V_3} \mathbb{1}_{B_2}) + \mathbb{E}(x^{V_3} \mathbb{1}_{A_3}) + \mathbb{E}(x^{V_3} \mathbb{1}_{B_3}) \\ &= \tfrac{2}{5} P_2(z_1, z_2) + \tfrac{2x+1}{5}\left[Q_1(z_1) + 2 Q_2(z_1, z_2)\right] \\ &\quad + \tfrac{2}{5} P_3(z_1, z_2, z_3) + \tfrac{2x+1}{5} Q_3(z_1, z_2, z_3) \end{aligned}$$

where $P_3(z_1, z_2, z_3) = z_3^3 - z_3 P_2(z_1, z_2)$,
$Q_3(z_1, z_2, z_3) = z_3^3 - z_3^2 Q_1(z_1) - 2 z_3 Q_2(z_1, z_2)$
with $z_i = g_3 \circ c_3^{i-1}(x) = \frac{x}{2} + \frac{i-1}{4}$, for $i = 1, 2, 3$.

More generally:

$$P_{k+1}(z_1,\ldots,z_{k+1}) = z_{k+1}^{k+1} - \sum_{\ell=2}^{k} \binom{k-1}{l-2} z_{k+1}^{k+1-l} P_\ell(z_1,\ldots,z_\ell).$$

$$Q_{k+1}(z_1,\ldots,z_{k+1}) = z_{k+1}^{k+1} - \sum_{\ell=1}^{k} \binom{k}{l-1} z_{k+1}^{k+1-l} Q_\ell(z_1,\ldots,z_\ell).$$

and we get the RDE for g_d: $z_{\ell,d}(x) = g_d\left(\frac{x}{2} + \frac{\ell-1}{2(d-1)}\right)$

$$g_d(x) = \frac{d-1}{2d-1} \sum_{k=2}^{d} \binom{d-2}{k-2} P_k(z_{1,d}(x),\ldots,z_{k,d}(x))$$
$$+ \left(\frac{d-1}{2d-1}x + \frac{1}{2d-1}\right) \sum_{k=2}^{d} \binom{d-1}{k-1} Q_k(z_{1,d}(x),\ldots,z_{k,d}(x))$$

List of Open Questions

Q1. Identify the recurrence/transience property of FM_3 and FM_4.

Q2. Show that $p_{d+1} < p_d$ for $FM(d, p)$.

Q3. Show that $FM(d, p') \preceq FM(d, p)$ if $p' \leq p$.

Q4. Is it possible to couple $FM(d_1, p_1)$ with $FM(d_2, p_2)$?

Q5. Show that $p_d \downarrow \frac{2-\sqrt{2}}{4}$ as $d \to \infty$.

Thanks!