Critical Branching Random Walks, Branching Capacity and Branching Interlacements

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Overview

Branching Random Walks

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$$\textbf{Branching Random Walk} \begin{cases} \textbf{Branching Process (GW-Process)} \\ \textbf{Random Walk} \end{cases}$$

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- At time 0, a particle is located at x.
- Suppose that, at each time n, a particle v is located at $\mathcal{S}_x(v)$.
- At time $n+1$, v gives birth to a random number, distributed according to μ, of children, and dies afterwards. Each child then moves to a new position $\mathcal{S}_x(v) + Y$, where Y is distributed according to θ. Different particles behave independently.

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In addition, we need some finite moment assumptions on μ, θ.

Background I: Global and Local

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What about Branching random walk?

- **Scaling Limit**: Super Brownian Motion, Brownian Snake.
- **Discrete Behavior**: Today's topic

Background II: Range of CBRW

Le Gall and Lin have established the following result about the range of critical branching random walk conditioned on the total number of offsprings being n, denoted by R_n:

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Le Gall and Lin have established the following result about the range of critical branching random walk conditioned on the total number of offsprings being n, denoted by R_n:

$$\frac{1}{n}R_n \xrightarrow{\mathrm{P}} c_1 \quad \text{as } n \to \infty, \text{ when } d \geq 5;$$

$$\frac{\log n}{n}R_n \xrightarrow{\mathrm{L}^2} c_2 \quad \text{as } n \to \infty, \text{ when } d = 4;$$

$$n^{-d/4}R_n \xrightarrow{\mathrm{d}} c_3\lambda_d(supp(\mathcal{I})) \quad \text{as } n \to \infty, \text{ when } d \leq 3;$$

where c_i are some constants and $\lambda_d(supp(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on $\mathbb{R}^d$ known as Integrated Super-Brownian Excursion.

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In the subcritical dimensions, they also establish the asymptotics of the hitting probability of a distant point by critical branching random walk:

$$\lim_{x \to \infty} \|x\|^2 \cdot P(\mathcal{S}_x \text{ visits } 0) = \frac{2(4-d)}{d\sigma^2}.$$

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- The asymptotic of $P(\mathcal{S}_x \text{ visits } 0)$ in other dimensions ($d \geq 4$);
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- The range of branching random walk with a general initial configuration.

We answer the first two questions.

We show that: when $d \geq 5$,

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and

$$\frac{\log n}{n} R_n \xrightarrow{\mathrm{P}} \frac{16\pi^2 \sqrt{\det Q}}{\sigma^2}.$$

Summary

$$P(\mathcal{S}_x \text{ visits } K) \sim \begin{cases} a_d \mathrm{BCap}(K)/\|x\|^{d-2}; & \text{when} \quad d \geq 5; \\ 1/2\sigma^2\|x\|^2 \log\|x\|; & \text{when} \quad d = 4; \\ 2(4-d)/d\sigma^2\|x\|^2; & \text{when} \quad d \leq 3. \end{cases}$$

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Note that

$$P(S_x \text{ visits } K) \sim \begin{cases} a_d \mathrm{Cap}(K)/\|x\|^{d-2}; & \text{when} \quad d \geq 3; \\ 1; & \text{when} \quad d \leq 2. \end{cases}$$

Branching Capacity: Intuition

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$$BCap(A) \approx How\ big\ A\ is\ with\ regard\ to\ CBRW$$

$$Cap(A) \approx How\ big\ A\ is\ with\ regard\ to\ RW$$

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Key Results

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Key Results

- BCap(A) is proportional to the asymptotics of the visiting probability of A by CBRW .
- We can construct a natural measure on the set of all 'infinite CBRW trajectories', such that BCap(A) is the measure of the set of all trajectories that intersects A.

Remark

- *The above results (and many others) are also true for RW and Capacity.*
- *Use the measure above as the intensity measure of PPP, we get the model of 'branching/random interlacements'.*

Capacity for Random Walk

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For any finite subset K of $\mathbb{Z}^d, d \geq 3$, the escape probability $\mathrm{ES}_K(x)$ is defined to be the probability that a random walk starting from $x \in \mathbb{Z}^d$, denoted by $S_x = (S_x(n))_{n\in\mathbb{N}}$, never returns to K.

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Theorem

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where τ_K is the first visiting time of K in a Depth-First search and a_d is the same constant as in the random walk case.

Definition of Branching Capacity

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To define the corresponding escape probabilities, we need to introduce adjusted versions of **infinite** branching random walk.

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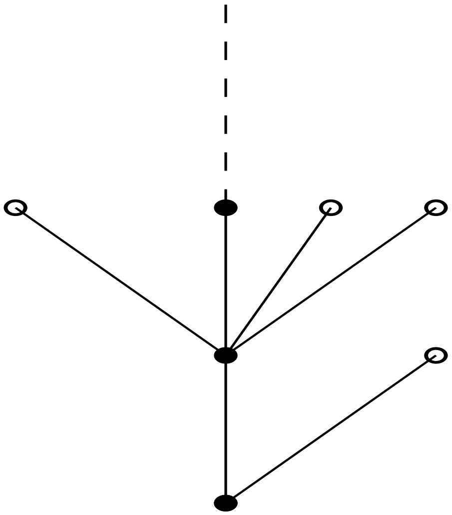
the GW-tree conditioned on survival



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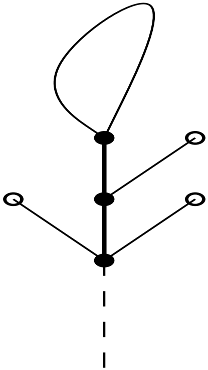

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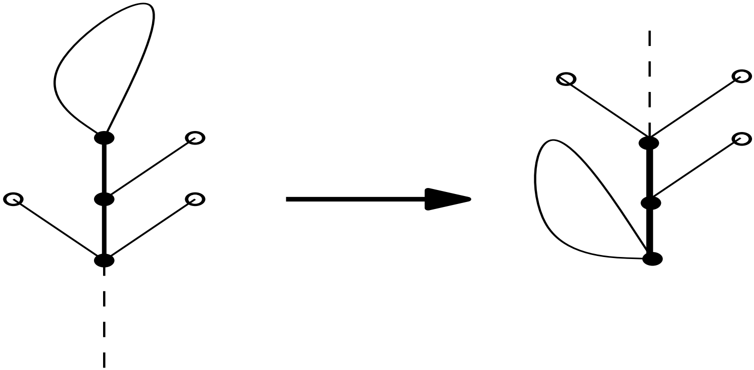

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T_{v} and the local limit of GW-trees around any vertex

Theorem (Aldous)

The local limit of GW-trees (conditioned on the total size) around a uniformly selected vertex is T_{v}.

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Theorem

The local limit of GW-trees (conditioned on the total size) around any prefixed vertex is T_{v}*, as long as this vertex is not too close to the root.*

Invariance Property of T_{v}

KEY Property

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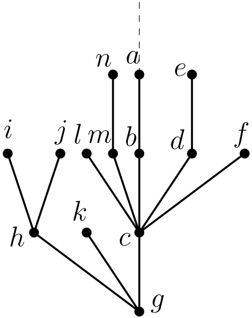
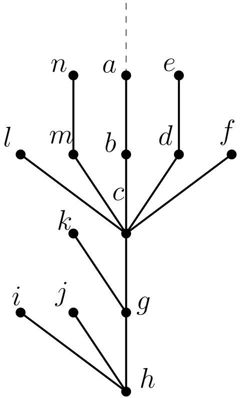
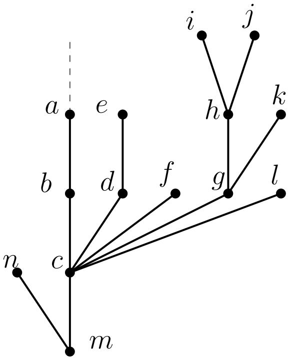
where $\tau_{\mathrm{v}}(\mathrm{t})$ is t re-rooted at the next vertex.

Re-root a tree at different vertices



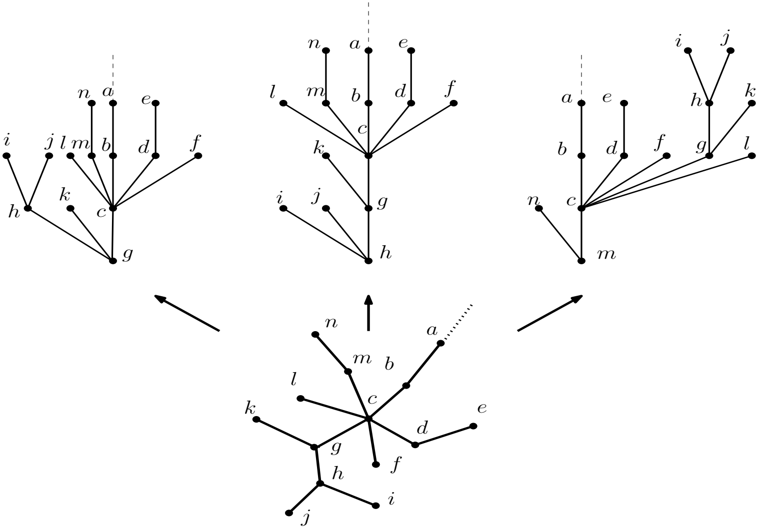


Re-rooted at h



Re-rooted at m

Re-root a tree at different vertices



Definition of escape probability and branching capacity

Now, we can define branching capacity for any finite subset of $\mathbb{Z}^d$,

Definition

$$\mathrm{Es}_A(x) = P[\mathcal{S}_x^{\mathrm{v}}(\text{all vertices before the root}) \cap A = \emptyset];$$

$$\mathrm{Esc}_A(x) = P[\mathcal{S}_x^{\mathrm{v}}(\text{all vertices after the root}) \cap A = \emptyset].$$

Then:

$$\mathrm{BCap}(A) = \sum_{a \in A} \mathrm{Esc}_A(a) = \sum_{a \in A} \mathrm{Es}_A(a).$$

Now, we can define capacity for any finite subset of $\mathbb{Z}^d$,

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Then:

$$\mathrm{Cap}(A) = \sum_{a \in A} \mathrm{ES}_A^-(a) = \sum_{a \in A} \mathrm{ES}_A(a).$$

Branching capacity and visiting probability

Theorem

For any nonempty finite subset K of $\mathbb{Z}^d$, $d \geq 5$ and $a \in K$, we have

$$\lim_{x \to \infty} \|x\|^{d-2} \cdot P(\mathcal{S}_x \text{ visits } K) = a_d BCap(K);$$

$$\lim_{x \to \infty} P(\mathcal{S}_x(\tau_K) = a | \mathcal{S}_x \text{ visits } K) = Es_K(a)/BCap(K).$$

Moreover, we have

$$\lim_{x \to \infty} \overline{\Theta}_x \overset{d}{=} \mathrm{m}_K,$$

where $\overline{\Theta}_x$ is (a random point measure) the conditional entering measure of K by $\mathcal{S}_x$, conditioned on $\mathcal{S}_x$ visits K.

We also have:

Theorem

For any finite $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ with $\rho(x, A) \geq 0.1 diam(A)$, we have:

$$P(\mathcal{S}_x \text{ visits } A) \asymp \frac{BCap(A)}{(\rho(x, A))^{d-2}},$$

where $f(x, A) \asymp g(x, A)$ indicates that there exists positive constants c_1, c_2 independent of x, A such that $c_1 f(x, A) \leq g(x, A) \leq c_2 f(x, A)$.

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Note that for random walk:

$$P(S_x \text{ visits } A) \asymp \frac{\mathrm{Cap}(A)}{(\rho(x, A))^{d-2}}.$$

Branching capacity of balls

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Theorem

Let $B^m(r)$ be the m-dimensional balls with radius r (as a subset of $\mathbb{Z}^d$), i.e. $\{z = (z_1, 0) \in \mathbb{Z}^m \times \mathbb{Z}^{d-m} = \mathbb{Z}^d : |z_1| \le r\}$, then

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$$BCap(B^m(r)) \asymp \begin{cases} r^{d-4} & \text{if} \quad m \geq d-3; \\ r^{d-4}/\log r & \text{if} \quad m = d-4; \\ r^m & \text{if} \quad m \leq d-5. \end{cases}$$

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One might compare this with the corresponding result about regular capacity:

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We believe:

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Theorem

Suppose that μ_1, μ_2 are two nondegenerate critical offspring distributions with finite second moment. Then, there is a $C = C(\mu_1, \mu_2) > 0$ such that for all finite $A \subseteq \mathbb{Z}^d$,

$$C^{-1} \cdot BCap_{\mu_1,\theta}(A) \leq BCap_{\mu_2,\theta}(A) \leq C \cdot BCap_{\mu_1,\theta}(A).$$

Proof of $\mathrm{BCap}_{\mu_1}(A) \asymp \mathrm{BCap}_{\mu_2}(A)$

In fact, we will show:

$$P_{\mu_1}(\mathcal{S}_x \text{ visits } A) \asymp P_{\mu_2}(\mathcal{S}_x \text{ visits } A).$$

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Recursive formula for $P(\mathcal{S}_x \text{ visits } A)$:
Let $f(t) = 1 - \sum_{k \geq 0} \mu(k)(1-t)^k$.

$$u_0(x) = 1_A(x),\ u_n(x) = 1, \forall x \in A;\ u_{n+1}(x) = f(\mathcal{A}u_n(x))\,\forall x \notin A,$$

where $\mathcal{A}u(x) = \sum_{z \in \mathbb{Z}^d} \theta(z) u(x+z)$. Then

$$P(\mathcal{S}_x \text{ visits } A) = \lim_{n \to \infty} u_n(x).$$

Proof of $\mathrm{BCap}_{\mu_1}(A) \asymp \mathrm{BCap}_{\mu_2}(A)$

Let $f_i(t) = 1 - \sum_{k\geq 0} \mu_i(k)(1-t)^k \ (i = 1, 2)$

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Lemma

There exists $C = C(\mu_1, \mu_2) > 1$ such that, for all $t \in [0, 1]$,

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Therefore, we have $u_\infty^1(x) \leq Cu_\infty^2(x)$, i.e.

$$P_{\mu_1}(\mathcal{S}_x \text{ visits } A) \leq CP_{\mu_2}(\mathcal{S}_x \text{ visits } A).$$

The model of random interlacement consists of a countable collection of doubly infinite random walk trajectories on $\mathbb{Z}^d, d \geq 3$. This model is constructed via a Poisson point process with intensity measure $u\nu_0$. The measure ν_0 is supported on the space of doubly infinite nearest-neighbour trajectories modulo time shifts on $\mathbb{Z}^d$. The union of the range of trajectories contained in the support of this Poisson point process defines the random subset I^u of $\mathbb{Z}^d$, called the random interlacement at level u.

The model of branching interlacement consists of a countable collection of infinite tree-indexed random walk trajectories on $\mathbb{Z}^d, d \geq 5$. This model is constructed via a Poisson point process with intensity measure $u\nu$. The measure ν is supported on the space of infinite tree-indexed random walk trajectories modulo vertex shifts on $\mathbb{Z}^d$. The union of the range of trajectories contained in the support of this Poisson point process defines the random subset $\mathcal{I}^u$ of $\mathbb{Z}^d$, called the branching interlacement at level u.

Characterization of branching interlacements

The law of the branching interlacement at level u can be characterized as the unique distribution on $\{0,1\}^{\mathbb{Z}^d}$ such that

Characterization

$$P[\mathcal{I}^u \cap K = \emptyset] = \exp(-u \cdot \mathrm{BCap}(K)), \text{ for every } K \subset\subset \mathbb{Z}^d, \tag{1}$$

where $\mathrm{BCap}(K)$ is the branching capacity of K.

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Note that (1) uniquely determines the law of $\mathcal{I}^u$ by the inclusion-exclusion principle:

$$\begin{aligned} P[\mathcal{I}^u \cap K = A] &= \sum_{B \subseteq A} (-1)^{|A \setminus B|} P[\mathcal{I}^u \cap (K \setminus B) = \emptyset] \\ &= \sum_{B \subseteq A} (-1)^{|A \setminus B|} \exp(-u\mathrm{BCap}(K \setminus B)). \end{aligned}$$

Characterization II

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Let N_K be a Poisson random variable with parameter $u \cdot \mathrm{BCap}(K)$, and $(X^j)_{j\geq 1}$ be i.i.d. branching random walks with m_K, the 'branching harmonic measure' from infinity of K as the initial measure.

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Recall

$$P[\mathcal{S}_x \text{ visits } K] \sim \frac{a_d \mathrm{BCap}(K)}{\|x\|^{d-2}}; \quad \lim_{x\to\infty} \overline{\Theta}_x \overset{\mathrm{d}}{=} \mathrm{m}_K$$

where $\overline{\Theta}_x$ is the conditional entering measure of K by $\mathcal{S}_x$, conditioned on $\mathcal{S}_x$ visits K.

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Remark

We can regard branching interlacement as the trajectory of the BRWs form Infinity.

Constructive definition of Branching interlacements

$$W = \{(\mathrm{t}, \mathcal{S}) : \mathrm{t} \in \mathbb{T}_\infty, \mathcal{S} : \mathrm{t} \to \mathbb{Z}^d\};$$

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τ is the transformation on W induced by τ_{v}.

$$w_1 \sim w_2 \text{ iff } \exists n \in \mathbb{Z}\,; w_1 = \tau^{(n)}(w_2);$$

$$W^* = W/\sim;$$

$$W_K = \{w \in W : \exists n \in \mathbb{Z}, w(n) \in K\};$$

$$H_K(w) = \inf\{n : w(n) \in K\};$$

$$W_K^0 = \{w \in W_K : H_K(w) = 0\}.$$

Construction of the intensity measure

Recall that we have an invariant measure T_{v} and the corresponding vertex shift τ_{v}

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- Each vertex, except the root, is normal or special.
- The root produces i individuals with probability $\mu(i-1)$. The last individual is special; the others (if any) are normal.
- A normal vertex produces only normal individuals, independently, according to μ.
- A special vertex produces individuals independently, according to the size-biased distribution $\tilde{\mu}$, given by $\tilde{\mu}(i) = i\mu(i)$. One of them, chosen uniformly at random, is special; the others (if any) are normal.

Construction of the intensity measure

For any $K \subset\subset \mathbb{Z}^d$, denote by Q_K, the finite measure on W_K such that

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Proposition

For any measurable set $A \subset W$, we have

$$\sum_{x \in \mathbb{Z}^d} P_x^{\mathrm{v}}[A] = \sum_{x \in \mathbb{Z}^d} P_x^{\mathrm{v}}[\tau(A)].$$

KEY FEATURE

$$Q_K[\pi^{-1}(A) \cap W_K] = Q_{K'}[\pi^{-1}(A) \cap W_K], \quad \forall A \subset W^*.$$

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Definition of the intensity measure

$$\nu[A] = \lim_{K \uparrow \mathbb{Z}^d} Q_K[\pi^{-1}(A)], \quad \forall A \subset W^*.$$

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Definition

The branching interlacement at level u is defined to be

$$\mathcal{I}^u = \mathcal{I}^u(\omega) = \bigcup_{n \geq 0} \mathrm{Range}(\overline{w}_n), \text{where} \quad \omega = \sum_{n \geq 0} \delta_{\overline{w}_n},$$

is distributed as the PPP with intensity measure $u\nu$.

The local limit of branching random walks in tori

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For any K, finite subset of $\mathbb{Z}^d$, we have

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More precisely,

$$\lim_{N\to\infty} P[\mathrm{Range}(\mathcal{S}_N^n) \cap K = A] = P[\mathcal{I}^u \cap K = A], \forall A \subset K.$$

The range of branching random walks

Recall that we have

$$\frac{R_n}{n} \xrightarrow{\mathrm{P}} c_1, \text{ when } d \geq 5,$$

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Note that, for random walk, we have:

$$\frac{R_n}{n} \xrightarrow{\mathrm{P}} \mathrm{Cap}(\{0\}), \quad \text{when } d \geq 3.$$

Cover times by branching random walks

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Answer

The cover time of $\mathbb{T}_N$ is concentrated at $N^d \log N^d / \mathrm{BCap}(\{0\})$.

Cover times by branching random walks

Theorem

Let $n = n(N)$ be an integer-valued function of N. For any $\epsilon \in (0,1)$, we have if $n(N) > (1+\epsilon)N^d \log N^d / BCap(\{0\})$, then

$$\lim_{N\to\infty} P[\mathrm{Range}(\mathcal{S}_N^n) = \mathbb{T}_N] = 1;$$

if $n(N) < (1-\epsilon)N^d \log N^d / BCap(\{0\})$, then

$$\lim_{N\to\infty} P[\mathrm{Range}(\mathcal{S}_N^n) = \mathbb{T}_N] = 0.$$

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A subset $K \subset \mathbb{Z}^d$ is called recurrent if

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Wiener's Test says that:
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$$K \text{ is recurrent } \Leftrightarrow \sum_{n=1}^{\infty} \frac{\text{Cap}(K_n)}{2^{n(d-2)}} = \infty.$$

Definition

Let K be a subset of $\mathbb{Z}^d$ ($d \geq 5$). We call K a branching recurrent (B-recurrent) set if

$$P(\mathcal{S}_0^\infty \text{ visits } K \text{ infinitely often}) = 1,$$

and a branching transient (B-transient) set if

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$$K \text{ is } \mathrm{B-recurrent} \Leftrightarrow \sum_{n=1}^{\infty} \frac{BCap(K_n)}{2^{n(d-4)}} = \infty.$$

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Therefore, K is B-recurrent.

Branching recurrence of $\mathbb{Z}^4$

By projection to $\mathbb{Z}^4$, we get that 0 is visited infinitely often by the incipient infinite snake, almost surely.

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Corollary

The incipient infinite snake in $\mathbb{Z}^4$ almost surely visits any vertex infinitely often.

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Corollary

K is B-recurrent w.r.t. (μ_1, θ) *if and only if K is B-recurrent w.r.t.* (μ_2, θ).

Variational formula for branching capacity and branching harmonic measure

How to characterize $\text{BCap}(A)$ and m_A?
Note that for random walk capacity we have:

Variational formula for random walk capacity

$$\text{Cap}(A) = \left(\inf \int\int g(x,y)\nu(dx)\nu(dy)\right)^{-1},$$

where inf is over all probability measures on A, and $g(x,y)$ is the discrete Green function.

Branching Capacity is a $(d-4)$-Capacity?

Conjecture

$$\mathrm{BCap}(A) \asymp \left(\inf \int \int (|x-y| \vee 1)^{4-d} \nu(dx)\nu(dy) \right)^{-1},$$

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where inf is over all probability measures on A.

If so, we have

$$\text{BCap}_{\mu,\theta_1}(\bullet) \asymp \text{BCap}_{\mu,\theta_2}(\bullet)$$

Harnack inequality for CBRW

Harnack Inequality for random walk

Suppose $U \subset \mathbb{R}^d$ is open and connected, and K is a compact subset of U. Then there exist $C = C(K, U)$ and $N = N(K, U)$ such that if $n \geq N$,

$$U = U_n = \{x \in \mathbb{Z}^d : n^{-1} \in U\}, \ K = K_n = \{x \in \mathbb{Z}^d : n^{-1} \in K\},$$

we have:

$$\mathrm{h}_U(x, a) \leq C\mathrm{h}_U(y, a), \quad x, y \in K, a \in \partial U,$$

where $\mathrm{h}_U(x, a) = P(S_x(\tau_{U^c}) = a)$ is the exit measure.

Harnack inequality for CBRW

Harnack Inequality for critical branching random walk (Conjecture)

Suppose $U \subset \mathbb{R}^d$ is open and connected, and K is a compact subset of U. Then there exist $C = C(K, U)$ and $N = N(K, U)$ such that if $n \geq N$,

$$U = U_n = \{x \in \mathbb{Z}^d : n^{-1} \in U\}, \; K = K_n = \{x \in \mathbb{Z}^d : n^{-1} \in K\},$$

we have:

$$\mathrm{h}_U(x, a) \leq C\mathrm{h}_U(y, a), \quad x, y \in K, a \in \mathcal{M}(\partial U),$$

where $\mathrm{h}_U(x, a) = P(\Theta_{U^c}(x) = a)$ is the exit measure.

Coupling between Branching interlacements and tree-indexed random walks in tori

For random walks and random interlacements, we have the following coupling result.

Theorem (Teixeria and Windisch)

For any $u > 0, \alpha > 0, \epsilon \in (0,1)$, there exists a constant c depending on d, u, α, ϵ and a coupling $(\Omega, \mathcal{A}, Q)$ of $S = S_{[0,uN^d]}$ with random interlacements $I^{u(1-\epsilon)}$ and $I^{u(1+\epsilon)}$ on $\mathbb{Z}^d$, such that

$$Q[I^{u(1-\epsilon)} \cap A \subseteq S \cap A \subseteq I^{u(1+\epsilon)} \cap A] \geq 1 - cN^{-\alpha}, \forall N > 1,$$

where A is a box with length $N^{1-\epsilon}$.

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Question

Can we show some analogous results for branching interlacements?

Cover times

Recall that for $d \geq 5$, the cover time of a d-dimensional torus by critical branching random walk is concentrated at $N^d \log N^d / \mathrm{BCap}(\{0\})$. What about other dimensions?

Cover times

Recall that for $d \geq 5$, the cover time of a d-dimensional torus by critical branching random walk is concentrated at $N^d \log N^d / \mathrm{BCap}(\{0\})$. What about other dimensions?For the cover time (denoted by C_N) of tori by random walks, we have:

- for $d \geq 3$, $\mathbb{E}[C_N] \sim N^d \log N^d / \mathrm{Cap}(\{0\})$. Moreover $C_N \mathrm{Cap}(\{0\})/N^d - \log N^d \xrightarrow{\mathrm{d}}$ Gumbel distribution(Belius);
- for $d = 2$, $\mathbb{E}[C_N] \sim 4N^2(\log N)^2/\pi$(Dembo-Peres-Rosen-Zeitouni);
- for $d = 1$, C_N/N^2 converges in law to the time needed for BM to cover the 1d-torus with unit length.

Question

Do we have the analogous results for the cover time by CBRW?

The End

Thanks