Conformal Invariance in 2D Lattice Models

Part 2: Random Cluster Model

Hao Wu (THU)

Part 1: Bernoulli Percolation
Part 2: Random Cluster Model
Part 3: Ising Model
## Bernoulli percolation vs. FK-percolation

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FK-percolation definition

Fortuin and Kasteleyn

FK-percolation: also called random-cluster model. It is a generalization of Bernoulli percolation where there is dependence between edges.

- \( G = (V, E) \) is a finite graph
- configuration \( \omega \in \{0, 1\}^E \), \( o(\omega) \), \( c(\omega) \), \( k(\omega) \)
- edge-parameter \( p \in [0, 1] \), cluster-parameter \( q > 0 \)

FK-percolation on \( G \) is the probability measure defined by

\[
\phi_{p,q,G}[\omega] \propto p^{o(\omega)}(1 - p)^{c(\omega)} q^{k(\omega)}.
\]
FK-percolation—boundary conditions

Fix a partition $\xi$ of $\partial G$, and identify the vertices in $\partial G$ that belong to the same component of $\xi$. FK-percolation on $G$ with parameters $(p, q)$ and boundary conditions $\xi$ is the probability measure:

$$\phi_{p,q,G}^\xi[\omega] \propto p^{o(\omega)}(1 - p)^{c(\omega)}q^{k(\omega,\xi)}.$$
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- wired-b.c.: $\phi_{p,q,G}^1$
- free-b.c.: $\phi_{p,q,G}^0$
- Dobrushin-b.c.
- induced by a config. outside $G$
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Domain Markov Property

Suppose $G' \subset G$, for any $\psi \in \{0, 1\}^{E(G) \setminus E(G')}$,

$$\phi_{p,q,G}^{\xi}[X | \omega_e = \psi_e, \forall e \in E(G) \setminus E(G')] = \phi_{p,q,G'}^{\psi \xi}[X].$$
Theorem (FKG Inequality)

Fix $p \in [0, 1]$, $q \geq 1$, a finite graph $G$ and some boundary conditions $\xi$. For any two increasing events $A$ and $B$, we have

$$\phi_{p,q,G}^\xi [A \cap B] \geq \phi_{p,q,G}^\xi [A] \phi_{p,q,G}^\xi [B].$$
Theorem (FKG Inequality)

Fix \( p \in [0, 1] \), \( q \geq 1 \), a finite graph \( G \) and some boundary conditions \( \xi \). For any two increasing events \( A \) and \( B \), we have

\[
\phi_{p,q,G}^\xi[A \cap B] \geq \phi_{p,q,G}^\xi[A]\phi_{p,q,G}^\xi[B].
\]

- Given two proba. measures \( \mu_1, \mu_2 \), we write \( \mu_1 \leq_{st} \mu_2 \), if \( \mu_1[A] \leq \mu_2[A] \) for all increasing event \( A \).
- A proba. measure \( \mu \) strictly positive if \( \mu[\omega] > 0 \) for all \( \omega \).

Theorem (Holley inequality)

Let \( \mu_1, \mu_2 \) be strictly positive probability measures on the finite state space such that

\[
\mu_2[\omega^e]\mu_1[\eta^e] \geq \mu_2[\omega^e]\mu_1[\eta^e], \quad \forall e \in E, \forall \eta \leq \omega.
\]

Then \( \mu_1 \leq_{st} \mu_2 \).
FKG Inequality: consequences

Corollary (Monotonicity)

Fix \( p \leq p' \) and \( q \geq 1 \), a finite graph \( G \) and some b.c. \( \xi \).
We have \( \phi_{p,q,G}^{\xi} \leq \phi_{p',q,G}^{\xi} \).

Corollary (Comparison between boundary conditions)

Fix \( p \in [0, 1] \) and \( q \geq 1 \), a finite graph \( G \). For any b.c. \( \xi \leq \psi \), we have \( \phi_{p,q,G}^{\xi} \leq \phi_{p,q,G}^{\psi} \).
In particular, for any b.c. \( \xi \), we have \( \phi_{p,q,G}^{0} \leq \phi_{p,q,G}^{\xi} \leq \phi_{p,q,G}^{1} \).

Corollary (Finite-energy property)

Fix \( p \in [0, 1] \) and \( q \geq 1 \), a finite graph \( G \), and some b.c. \( \xi \), we have

\[
\frac{p}{p + (1 - p)q} \leq \phi_{p,q,G}^{\xi} [\omega(f) = 1 \mid \omega(e) = \psi(e) \ \forall \ e \in E(G) \setminus \{f\}] \leq p.
\]
Bernoulli percolation vs. FK-percolation

Bernoulli percolation
Independent percolation
- FKG inequality
- Phase transition
- Critical value: $p_c = p_{sd}$
- Subcritical: exp. decay
- Continuity of PT

FK percolation
dependent percolation
- True for $q \geq 1$
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- True for $q \geq 1$
- True for $1 \leq q \leq 4$
- False for $q > 4$
Infinite volume measure

Let $\xi_n$ be a sequence of b.c. The sequence $\phi_{p,q,\Lambda_n}^{\xi_n}$ is said to converge to the infinite-volume measure $\phi_{p,q}$ if

$$\lim_{n \to \infty} \phi_{p,q,\Lambda_n}^{\xi_n}[A] = \phi_{p,q}[A],$$

for any event $A$ depending only on the status of finitely many edges.

**Proposition**

Fix $p \in [0, 1]$ and $q \geq 1$. There exist two (possibly equal) infinite-volume random-cluster measures $\phi_{p,q}^0$ and $\phi_{p,q}^1$ such that for any event $A$ depending on a finite number of edges,

$$\lim_{n \to \infty} \phi_{p,q,\Lambda_n}^1[A] = \phi_{p,q}^1[A], \quad \lim_{n \to \infty} \phi_{p,q,\Lambda_n}^0[A] = \phi_{p,q}^0[A].$$
Ergodicity

Lemma

Fix \( q \geq 1 \). The infinite-volume measures \( \phi_{p,q}^0 \) and \( \phi_{p,q}^1 \) are translation invariant and are ergodic.
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Fix $q \geq 1$. For $\phi_{p,q}^0$ or $\phi_{p,q}^1$, either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.
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The measures $\phi_{p,q}^0$ and $\phi_{p,q}^1$ are extremal:

$$\phi_{p,q}^0 \leq \text{st } \phi_{p,q} \leq \text{st } \phi_{p,q}^1.$$
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$$\phi_{p,q}^0 \leq \text{st } \phi_{p,q} \leq \text{st } \phi_{p,q}^1.$$

Question

Do we have $\phi_{p,q}^0 = \phi_{p,q}^1$?
### Bernoulli percolation vs. FK-percolation

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**Theorem**

Fix $q \geq 1$. There exists a critical point $p_c = p_c(q) \in [0, 1]$ such that

- For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely.
- For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.

**Lemma**

Fix $q \geq 1$. we have $\phi^0_{p,q} = \phi^1_{p,q}$ for all but countably many values of $p$. 
Bernoulli percolation vs. FK-percolation

Bernoulli percolation
Independent percolation

- FKG inequality
- Phase transition
- Critical value: $p_c = p_{sd}$
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FK percolation
dependent percolation

- True for $q \geq 1$ ☑
- $q \geq 1$ : $\infty$-volume measure ☑
- $q \geq 1$ : Phase transition ☑
- True for $q \geq 1$.
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- False for $q > 4$
Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$. The critical value $p_c$ is given by

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$ 

Proposition

The dual configuration of the random-cluster model on $G$ with parameters $(p, q)$ and b.c. $\xi$ is the random-cluster model with parameters $(p^*, q)$ on $G^*$ with b.c. $\xi^*$ where $p^* = p^*(p, q)$ satisfying

$$\frac{pp^*}{(1 - p)(1 - p^*)} = q.$$
Lemma

Fix $q \geq 1$, we have

$$
\phi_{p_{sd}(q),q}^0[0 \leftrightarrow \infty] = 0.
$$

Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$.

- If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$,
  $$
  \phi_{p,q,\Lambda_n}^1[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.
  $$

- If $p > p_c$, then there exists $C > 0$ such that
  $$
  \phi_{p,q}^1[0 \longleftrightarrow \infty] \geq C(p - p_c).
  $$
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Continuity of the phase transition

Theorem

- **Fix** $1 \leq q \leq 4$, we have

  $$\phi_{p_c,q}^1[0 \leftrightarrow \infty] = 0.$$ 

- **Fix** $q > 4$, we have

  $$\phi_{p_c,q}^1[0 \leftrightarrow \infty] > 0, \quad \phi_{p_c,q}^0[0 \leftrightarrow \infty] = 0.$$ 

Consequence

- When $1 \leq q \leq 4$, we have $\phi^1 = \phi^0$, and continuous PT.
- When $q > 4$, we have $\phi_{p_c,q}^1 \neq \phi_{p_c,q}^0$, and discontinuous PT for $\phi_{p_c,q}^1$. 
Bernoulli percolation vs. FK-percolation

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FK-Ising model

FK-Ising model: random-cluster model with $q = 2$.

$L = (\mathbb{Z}^2, E(\mathbb{Z}^2))$: the square lattice; $L^*$: the dual lattice
$L\diamondsuit$: the medial lattice.

- vertices: the centers of edges of $L$.
- edges: connecting nearest neighbors.

$L_\delta = \sqrt{2}\delta L, L^*_\delta, L_{\diamondsuit\delta}$. The mesh-size of $L_{\diamondsuit\delta}$ is $\delta$. 

(a) The square lattice.  (b) The dual lattice.  (c) The medial lattice.
Dobrushin domain

For a simply connected domain $\Omega$, we set $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$.

For a Dobrushin domain $(\Omega; a, b)$, let $(\Omega^\diamond; a_\delta, b_\delta)$ be an approximation. Dobrushin b.c. : edges of $(ba)$ are open (edges of $(ba)$ are wired), edges of $(a^* b^*)$ are dual-open (edges of $(ab)$ are free).
Loop representation

Fix a Dobrushin domain \((\Omega; a, b)\) with Dobrushin b.c. Draw self-avoiding loops on \(\Omega^\circ\) as follows: a loop arriving at a vertex of the medial lattice always makes a \(\pm \pi/2\) turn so as not to cross the open or dual open edges through this vertex.
FK fermionic observable

Definition

The edge FK fermionic observable is defined on edges of $\Omega_\delta^\diamondsuit$ by

$$F(\Omega_\delta^\diamondsuit; a_\delta^\diamondsuit, b_\delta^\diamondsuit)(e) = \mathbb{E}(\Omega_\delta^\diamondsuit; a_\delta^\diamondsuit, b_\delta^\diamondsuit) \left[ 1_{\{e \in \gamma\}} \exp \left( \frac{i}{2} W_\gamma(e, b_\delta) \right) \right],$$

where $W_\gamma(e, b_\delta)$ denotes the winding between the center of $e$ and $b_\delta^\diamondsuit$. The vertex FK fermionic observable is defined on vertices of $\Omega_\delta^\diamondsuit \backslash \partial \Omega_\delta^\diamondsuit$ by

$$F(\Omega_\delta^\diamondsuit; a_\delta^\diamondsuit, b_\delta^\diamondsuit)(v) = \frac{1}{2} \sum_{e \sim v} F(\Omega_\delta^\diamondsuit; a_\delta^\diamondsuit, b_\delta^\diamondsuit)(e),$$

where the sum is over the four medial edges having $v$ as an endpoint.
Conformal invariance

Theorem

Fix a Dobrushin domain \((\Omega; a, b)\). Consider the critical FK-Ising model. Let \(F_\delta\) be the vertex fermionic observable in \((\Omega^\wedge_\delta; a^\wedge_\delta, b^\wedge_\delta)\). Then, we have

\[
\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly},
\]

where \(\phi\) is any conformal map from \(\Omega\) on to the strip \(\mathbb{R} \times (0, 1)\) sending \(a\) to \(-\infty\) and \(b\) to \(+\infty\).
Discrete complex analysis

\( \mathbb{L} = (\mathbb{Z}^2, E(\mathbb{Z}^2)) : \) the square lattice; \( \mathbb{L}^* : \) the dual lattice
\( \mathbb{L}^\diamond : \) the medial lattice.

- vertices: the centers of edges of \( \mathbb{L} \).
- edges: connecting nearest neighbors.

\( \mathbb{L}_\delta = \sqrt{2}\delta \mathbb{L}, \mathbb{L}^*_\delta, \mathbb{L}^\diamond_\delta. \) The mesh-size of \( \mathbb{L}^\diamond_\delta \) is \( \delta \).
For \( x \in \mathbb{L}_\delta \) and \( h : \mathbb{L}_\delta \rightarrow \mathbb{C} \), define

\[
\Delta_\delta h(x) = \frac{1}{4} \sum_{y : y \sim x} (h(y) - h(x)).
\]

- \( h : \Omega_\delta \rightarrow \mathbb{C} \) is preharmonic if \( \Delta_\delta h(x) = 0, \forall x \in \Omega_\delta \).
- \( h : \Omega_\delta \rightarrow \mathbb{C} \) is pre-superharmonic if \( \Delta_\delta h(x) \leq 0, \forall x \in \Omega_\delta \).
- \( h : \Omega_\delta \rightarrow \mathbb{C} \) is pre-subharmonic if \( \Delta_\delta h(x) \geq 0, \forall x \in \Omega_\delta \).

The classical relation between preharmonic function and SRW:
Let \((X_n)\) be a SRW on \( \mathbb{L}_\delta \) killed at the first time it exits \( \Omega_\delta \), then \( h \) is preharmonic if and only if \((h(X_n))\) is a martingale.
Theorem

Let \((\Omega; a, b)\) be a Dobrushin domain,

let \(f\) be a bounded continuous function on \(\partial \Omega \setminus \{a, b\}\),

let \(h\) be the unique harmonic function on \(\Omega\), continuous on \(\overline{\Omega} \setminus \{a, b\}\), satisfying \(h = f\) on \(\partial \Omega \setminus \{a, b\}\).

Let \((\Omega_\delta; a_\delta, b_\delta)\) be a sequence of discrete Dobrushin domains converging to \((\Omega; a, b)\) in the Carathéodory sense.

Let \(f_\delta : \partial \Omega_\delta \to \mathbb{C}\) be a sequence of uniformly bounded functions converging to \(f\) uniformly away from \(a\) and \(b\).

Let \(h_\delta\) be the unique preharmonic function on \(\Omega_\delta\) such that \(h_\delta = f_\delta\) on \(\partial \Omega_\delta\).

Then \(h_\delta \to h\) locally uniformly as \(\delta \to 0\).
Preholomorphic function

For a function $f : \mathbb{L}_\delta \rightarrow \mathbb{C}$, and $x \in \mathbb{L}_\delta^*$, define

$$\bar{\partial}_\delta f(x) = \frac{1}{2}(f(E) - f(W)) + \frac{i}{2}(f(N) - f(S)),$$

where $N, E, S, W$ are the four vertices of $\mathbb{L}_\delta$ adjacent to $x$ indexed in the obvious way.

A function $f : \Omega_\delta \rightarrow \mathbb{C}$ is preholomorphic if $\bar{\partial}_\delta f(x) = 0$ for all $x \in \Omega_\delta^*$.

The equation $\bar{\partial}_\delta f(x) = 0$ is called the Cauchy-Riemann equation at $x$.

1. Sums of preholomorphic functions are preholomorphic.
2. Discrete contour integrals vanish in simply connected domain.
3. The primitive in simply connected domain is well-defined.
4. If a family $(f_\delta)$ of preholomorphic functions on $\Omega_\delta$ converges locally uniformly to $f$ on $\Omega$, then $f$ is holomorphic.

Attention: the product of two preholomorphics is not preholomorphic.
Spin-holomorphic

For \( e \in E(\mathbb{L}^\circ) \), we give an orientation: counterclockwise around white faces.

For \( e \in E(\mathbb{L}^\circ) \), we associate a direction \( \ell(e) \):
as in the figure. In other words, \( \ell(e) \) has the same direction as \( \sqrt{\bar{e}} \).

A function \( f \) is s-holomorphic if for any edge \( e \) of \( \Omega_\delta^\circ \), we have

\[
P_{\ell(e)}[f(x)] = P_{\ell(e)}[f(y)],
\]

where \( x, y \) are the endpoints of \( e \) and \( P_{\ell} \) is the orthogonal projection on the direction \( \ell \).

Proposition

Any s-holomorphic function \( f : \Omega_\delta^\circ \to \mathbb{C} \) is preholomorphic on \( \Omega_\delta^\circ \).
Theorem

Let $\Omega$ be a simply connected domain. Suppose $f : \Omega^\diamond \to \mathbb{C}$ is an $s$-holomorphic function and $b_0 \in \Omega_\delta$. Then, there exists a unique function $H : \Omega_\delta \cup \Omega^*_\delta \to \mathbb{C}$ such that

$$H(b_0) = 1, \quad \text{and} \quad H(b) - H(w) = \delta |P_{\ell(e)}[f(x)]|^2 (= \delta |P_{\ell(e)}[f(y)]|^2),$$

for every edge $e = (x, y)$ on $\Omega^\diamond_\delta$ bordered by a black face $b \in \Omega_\delta$ and a white face $w \in \Omega^*_\delta$.

For two neighboring sites $b_1, b_2 \in \Omega_\delta$, with $v$ being the medial vertex at the center of $(b_1, b_2)$,

$$H(b_1) - H(b_2) = \frac{1}{2} \Im(f(v)^2(b_1 - b_2)).$$

The same relation also holds for vertices of $\Omega^*_\delta$. 
Proposition

Denote by $H^\bullet$ the restriction of $H$ to $\Omega_\delta$ (black faces) and by $H^\circ$ the restriction of $H$ to $\Omega^*_\delta$ (white faces). If $f$ is s-holomorphic, then $H^\bullet$ is subharmonic and $H^\circ$ is superharmonic.
For a simply connected domain $\Omega$, we set $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$.

For a Dobrushin domain $(\Omega; a, b)$, let $(\Omega^\diamond; a_\delta, b_\delta)$ be an approximation. Dobrushin b.c. : edges of $(ba)$ are open (edges of $(ba)$ are wired), edges of $(a^* b^*)$ are dual-open (edges of $(ab)$ are free).
Fix a Dobrushin domain $(\Omega; a, b)$ with Dobrushin b.c. Draw self-avoiding loops on $\Omega^\circ$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm \pi/2$ turn so as not to cross the open or dual open edges through this vertex.
FK fermionic observable

Definition
The edge FK fermionic observable is defined on edges of $\Omega_\delta^\diamond$ by

$$F(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)(e) = \mathbb{F}(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond) \left[ 1_{\{e \in \gamma\}} \exp \left( \frac{i}{2} W_\gamma(e, b_\delta) \right) \right],$$

where $W_\gamma(e, b_\delta)$ denotes the winding between the center of $e$ and $b_\delta^\diamond$. The vertex FK fermionic observable is defined on vertices of $\Omega_\delta^\diamond \setminus \partial \Omega_\delta^\diamond$ by

$$F(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)(v) = \frac{1}{2} \sum_{e \sim v} F(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)(e),$$

where the sum is over the four medial edges having $v$ as an endpoint.
**Theorem**

Fix a Dobrushin domain \((\Omega; a, b)\). Consider the critical FK-Ising model. Let \(F_\delta\) be the vertex fermionic observable in \((\Omega_\delta; a_\delta, b_\delta)\). Then, we have

\[
\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly},
\]

where \(\phi\) is any conformal map from \(\Omega\) on to the strip \(\mathbb{R} \times (0, 1)\) sending \(a\) to \(-\infty\) and \(b\) to \(+\infty\).
The observable is s-holomorphic

Lemma

Consider a medial vertex \( v \in \Omega_\delta^\circ \setminus \partial \Omega_\delta^\circ \). We have

\[
F_\delta(N) - F_\delta(S) = i (F_\delta(E) - F_\delta(W)),
\]

where \( N, E, S, W \) are the four adjacent edges indexed in clockwise order.

Lemma

The vertex fermionic observable \( F_\delta \) is s-holomorphic.
Let $A$ be the black face bordering $a_{\delta}$. Define $H_{\delta} : \Omega_{\delta} \cup \Omega_{\delta}^* \rightarrow \mathbb{R}$ such that

$$H(A) = 1, \quad \text{and} \quad H_{\delta}(B) - H_{\delta}(W) = |P_{\ell(e)}[F_{\delta}(x)]|^2 = |P_{\ell(e)}[F_{\delta}(y)]|^2,$$

for the medial edge $e = (x, y)$ bordered by a black face $B \in \Omega_{\delta}$ and a white face $W \in \Omega_{\delta}^*$.

**Lemma**

- **The subharmonic function** $H_{\delta}^\bullet$ **is equal to 1 on** $(ba)$, **and it converges to 0 on** $(ab)$ **uniformly away from** $a$ **and** $b$.

- **The superharmonic function** $H_{\delta}^\circ$ **is equal to 0 on** $(a^*b^*)$, **and it converges to 1 on** $(b^*a^*)$ **uniformly away from** $a$ **and** $b$. 
Proposition

The sequence $(H_\delta)_{\delta > 0}$ converges to $\mathcal{S}\phi$ locally uniformly.

Theorem

Fix a Dobrushin domain $(\Omega; a, b)$. Consider the critical FK-Ising model. Let $F_\delta$ be the vertex fermionic observable in $(\Omega_\delta; a_\delta, b_\delta)$. Then, we have

$$\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly},$$

where $\phi$ is any conformal map from $\Omega$ on to the strip $\mathbb{R} \times (0, 1)$ sending $a$ to $-\infty$ and $b$ to $+\infty$.

Corollary

The exploration path in FK-Ising with Dobrushin boundary conditions converges to $\text{SLE}_{16/3}$. (Lecture on Oct. 30th)