2020 · 9 · 23

$G=(V,E)$

$\omega \in \{\bullet, \circ\}^V$

Bernoulli site perco. on G:

$\mathbb{P}_p[\omega(x)=\bullet]=p$, $\quad \mathbb{P}_p[\omega(x)=\circ]=1-p$.

indept. for distinct vertices.

$G=$ triangular lattice $\mathbb{T}$

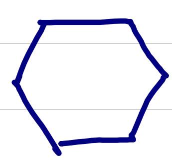
Phase transition

$\theta(p)=\mathbb{P}_p[0 \xleftrightarrow{B} \infty]$ increasing in p.

$p<p_c$, $\theta(p)=0$; $\quad p>p_c$, $\theta(p)>0$.

$p_c>0$. $\quad p_c<1$: duality.

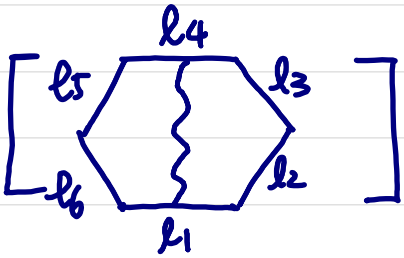
subcritical: $p<p_c$, $\mathbb{P}_p[0 \leftrightarrow \partial H_n] \leq e^{-cn}$.



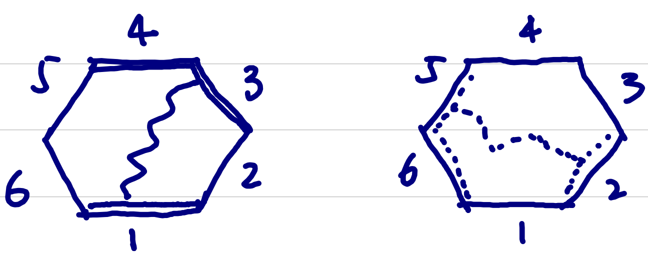
Thm. $p = \frac{1}{2}$. $\mathbb{P}_{1/2}[0 \longleftrightarrow \partial H_n] \le n^{-c}$.

thm. $\theta(\frac{1}{2}) = 0$. $p_c \ge \frac{1}{2}$. $p_c > \frac{1}{2}$ $\times$ $\Rightarrow p_c = \frac{1}{2}$.

$\boxed{\text{RSW estimate.}}$ $p = \frac{1}{2}$.

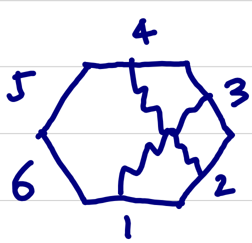
Lemma. $\mathbb{P}\Big[$  $\Big] \ge \frac{1}{8}$.

Pf:



$$\mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_3 \cup l_4] = \frac{1}{2}.$$

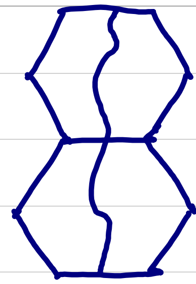
$$\underset{?}{\mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_3]} + \underset{\checkmark}{\mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_4]} \ge \frac{1}{2}.$$



$$\mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_4] \ge \mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_3,\ l_2 \overset{B}{\longleftrightarrow} l_4]$$

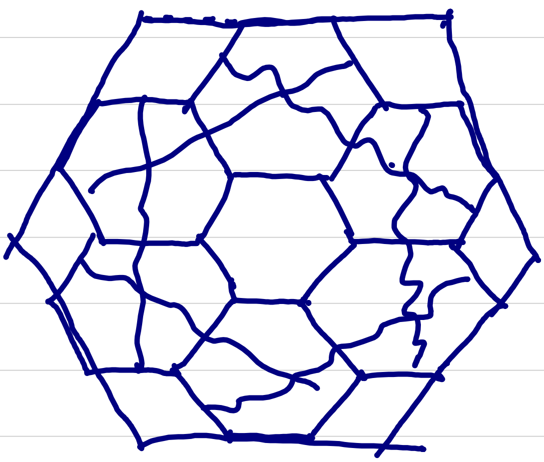
$$\ge \mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_3]^2$$

$$\mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_4]^{\frac{1}{2}} + \mathbb{P}[l_1 \overset{B}{\longleftrightarrow} l_4] \ge \frac{1}{2}.$$

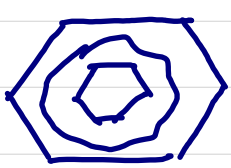
Lemma. $\mathbb{P}\Big[$  $\Big] \geq \frac{1}{18^2} > 0.$

c_1

cor.



$A_n := H_{3n} \setminus H_n$



$\mathbb{P}[\exists$ B circuit in $A_n] \geq c_2 > 0$

Pf. $\mathbb{P}[\exists$ B circuit in $A_n]$

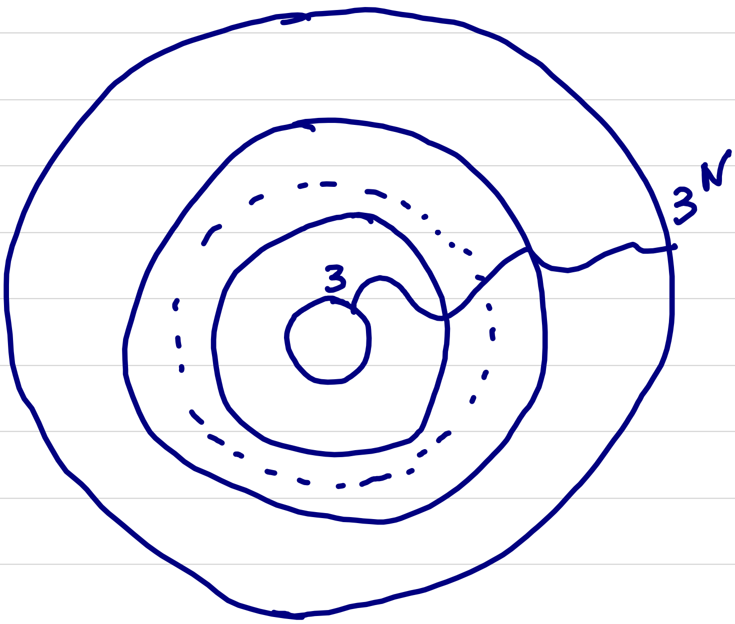
$\geq \mathbb{P}[$ six crossings $] \geq c_1^6 > 0.$

$c_2 = c_1^6.$

Pf of Thm.

$\mathbb{P}[\exists$ W circuit in $H_{3^m} \setminus H_{3^{m-1}}] \geq c > 0.$

$\mathbb{P}[\partial H_3 \overset{B}{\longleftrightarrow} \partial H_{3^N}] \leq (1-c)^N.$



$\mathbb{P}[0 \overset{B}{\longleftrightarrow} \partial H_n] \leq n^{-\alpha}, \quad \alpha > 0.$

$\theta(\frac{1}{2}) = 0.$

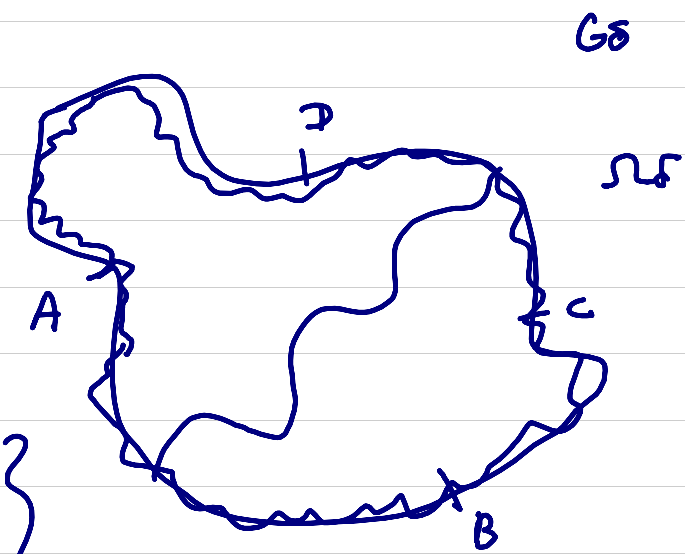
$p_c \geq \frac{1}{2}$. $\quad p_c > \frac{1}{2}$. exp. decay.

contradicts with RSW estimate

$p_c = \frac{1}{2}.$

G: honeycomb lattice $\quad G_\delta$

G^*: triangular. $\quad G^*_\delta$.



$(\Omega; A, B, C, D)$

$\mathcal{E}_\delta(\Omega; A, B, C, D)$

$= \left\{ \begin{array}{l} \exists \text{ B crossing in } \Omega_\delta \\ (A_\delta B_\delta) \text{ to } (C_\delta D_\delta) \end{array} \right\}.$

Thm.

- the proba. of $\mathcal{E}_\delta$ is convergent:

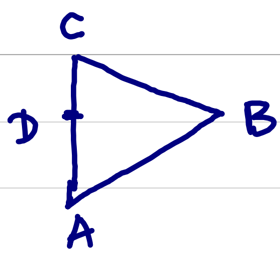
$$\mathbb{P}[\mathcal{E}_\delta(\Omega; A, B, C, D)] \to f(\Omega; A, B, C, D) \quad \text{as } \delta \to 0.$$

- f is conformal invariance: $\forall$ conformal map ϕ on Ω

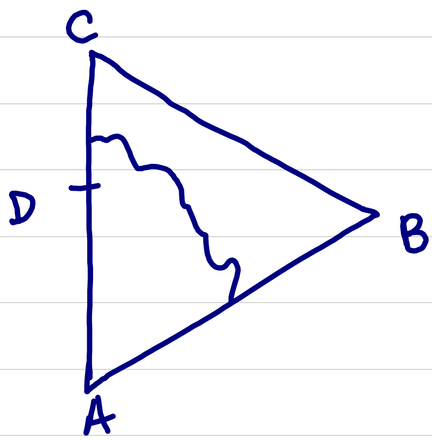
$$f(\phi(\Omega); \phi(A), \phi(B), \phi(C), \phi(D))$$
$$= f(\Omega; A, B, C, D).$$

-

$(\Omega; A, B, C, D) =$



$f = \dfrac{|CD|}{|CA|}$.



$(\Omega; A, B, C, D)$ quad. (topolgical rectangle).

Ω: non-empty bounded <u>simply connected domain</u>

$\Phi : \mathbb{U} \to \Omega$.

(Riemann Mapping Thm)

$\partial\Omega$: locally connected.

$\Updownarrow$

$\partial\Omega$ is a curve

$\Updownarrow$

Φ can be extended continuously to $\overline{\mathbb{U}}$.

$A, B, C, D \in \partial\Omega$. counterclockwise.

$(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta)$ $\quad G_\delta$.

$\rightarrow (\Omega; A, B, C, D)$ in Carathéodory sense.

$\exists$ conformal maps $\phi_\delta : \mathbb{U} \rightarrow \Omega_\delta$

$\phi : \mathbb{U} \rightarrow \Omega$.

$\phi_\delta \rightarrow \phi$ locally uniformly

$\phi_\delta^{-1}(A_\delta) \rightarrow \phi^{-1}(A)$

$\phi_\delta^{-1}(B_\delta) \rightarrow \phi^{-1}(B)$

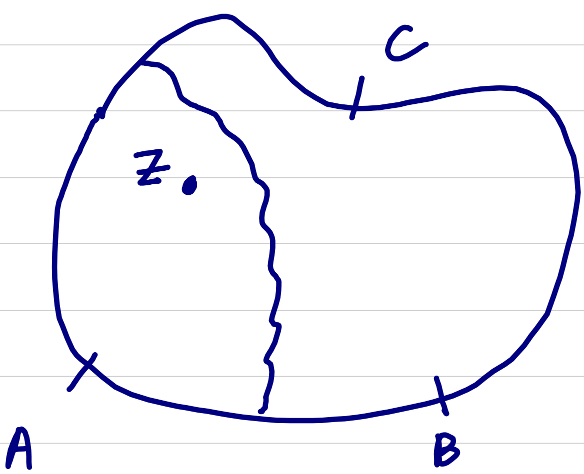
$\phi_\delta^{-1}(C_\delta) \rightarrow \phi^{-1}(C)$

$\phi_\delta^{-1}(D_\delta) \rightarrow \phi^{-1}(D)$.

$z \in \Omega$.

$E_A^\delta(z) = \{ \exists$ a B path disconnects A_δ, z_δ from $B_\delta, C_\delta \}$.

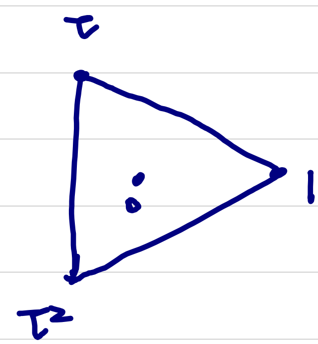
$H_A^\delta(z) = \mathbb{P}[E_A^\delta(z)]$.



$E_B^\delta(z)$, $E_C^\delta(z)$.

$H_B^\delta(z)$, $H_C^\delta(z)$.

$\tau = e^{\frac{2\pi i}{3}}$



$$H^\delta(z) := H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z)$$

$$S^\delta(z) := H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z).$$

step 1. $\{H_A^\delta, H_B^\delta, H_C^\delta\}_{\delta>0}$ tight / uniform cvg.

(RSW estimate $\Rightarrow$) $|H_A^\delta(x) - H_A^\delta(y)| \leq C|x-y|^\alpha$.

Step 2. Holomorphicity.

let H be any subsequential limit of H^δ
S S^δ.

H and S are holomorphic.

(Main contribution).

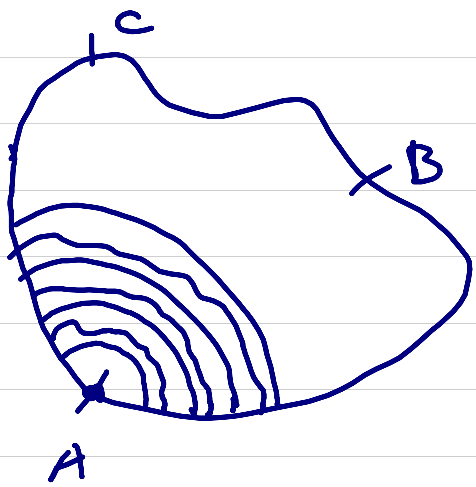
Step 3. H, S holomorphic Ω
bounded on Ω.
can be continuously extended to $\bar{\Omega}$.

they are uniquely determined by their boundary value.

S : holomorphic. real-valued.

S is contant. $S(A) = ?$

$S^\delta(z) = H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z)$,



$H_A^\delta(z)$

$H_A^\delta(A) \to 1 \ , \ \delta \to 0.$

$E_1, E_2, \dots, E_N,$ $\quad N = O\left(\frac{1}{\delta}\right)$

$\mathbb{P}[\, E_j \,] \geq c > 0.$

$$H_A^\delta(A) \geq \mathbb{P}\Big[\bigcup_1^N E_j\Big] \geq 1-(1-c)^N$$
$\to 1$ as $\delta \to 0.$

$$\mathbb{P}\Big[\bigcap_1^N E_j^c\Big] \leq (1-c)^N$$

$H_A^\delta(A) \to 1.$ $\quad H(A) = 1.$

$H_B^\delta(A) \to 0, \quad H_C^\delta(A) \to 0.$

$S(A) = 1.$

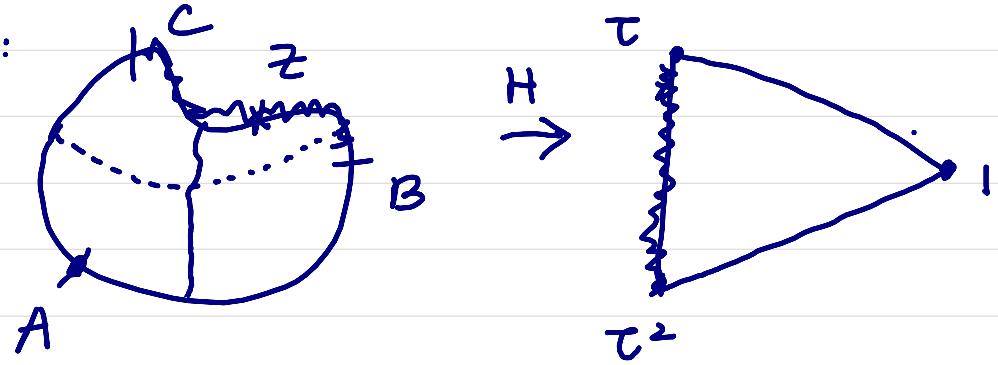
$S \equiv 1.$

H: $\quad H_A(A)=1, \quad H_B(A)=0, \quad H_C(A)=0.$

$H = H_A + \tau H_B + \tau^2 H_C.$

$H(A) = 1, \quad H(B) = \tau, \quad H(C) = \tau^2.$

Claim:



consider (BC). $\quad H(z) = ?$

$H_A(z) = 0. \quad H_B(z) + H_C(z) = 1$

$H_B(z)$ moves continuously from 1 to 0
as z moves from B to C.

H induces a one-to-one continuous map
from $[BC]$ to $[\tau\tau^2]$

$(CA), (AB).$

H · holomorphic in Ω

· extends continuously to $\bar{\Omega}$

· induces a continuous bijection from $\partial\Omega$ to $\partial\triangleright$

H is the conformal map from Ω onto $\triangleright$.

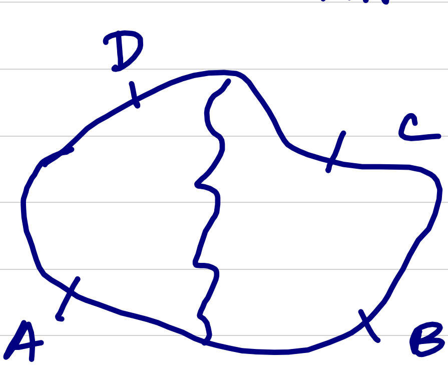
summary: $H_A + H_B + H_C = 1$

$H_A + \tau H_B + \tau^2 H_C = H$

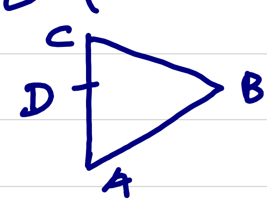
Re. $H_A + H_B + H_C = 1$

$H_A - \frac{1}{2} H_B - \frac{1}{2} H_C = \mathrm{Re} H$.

$H_A = \frac{1}{3}(1 + 2\,\mathrm{Re}\,H)$



$H_A(D) = \frac{1}{3}(1 + 2\,\mathrm{Re}\,H)$



A function $f : \Omega \to \mathbb{C}$.

holomorphic: $\forall z \in \Omega$, the limit exists

$$f'(z) = \lim_{|\varepsilon| \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}.$$

[Morera lemma] f holomorphic iff

$\forall$ simple closed smooth curve γ, $\oint_\gamma f = 0$.

$$H^\delta(z) = H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z).$$

H any sequential limit of H^δ.

γ_δ approximation of γ:

$(\gamma_\delta(k),\ 0 \le k \le N_\delta)$, $N_\delta = O\left(\frac{1}{\delta}\right)$.

$$I^\delta(\gamma) := \sum \frac{1}{2} \left(H^\delta(\gamma_\delta(k)) + H^\delta(\gamma(k+1))\right) \underline{(\gamma^\delta(k+1) - \gamma^\delta(k))}$$

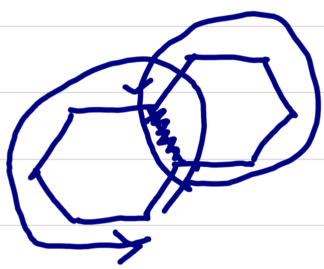
$\to \oint_\gamma H(z)\, dz$. Goal: $\oint_\gamma H = 0$.

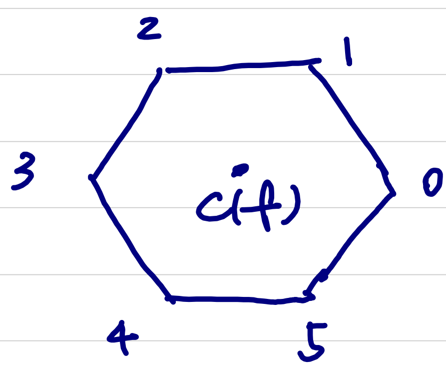
$I^\delta(\gamma) = o(1)$.

$e = (x,y)$. $\quad H^\delta(e) = \frac{1}{2}\left(H^\delta(x) + H^\delta(y)\right)$

$$\partial_e H^\delta = H^\delta(y) - H^\delta(x).$$

$$I^\delta(\gamma) = \sum_{e\in\gamma} e\, H^\delta(e) = \sum_{f\in\mathring{\gamma}} \underline{\sum_{e\in\partial f} e\, H^\delta(e)}$$





$$\sum_{e\in\partial f} e\, H^\delta(e) = \sum_0^5 (x_{k+1} - x_k)\frac{1}{2}\left(H^\delta(x_{k+1}) + H^\delta(x_k)\right)$$

"integrate by parts"

$$= \sum_0^5 (x_k - c(f))\frac{1}{2}\left(H(x_k) + H(x_{k-1})\right)$$

$$- \sum_0^5 (x_k - c(f))\frac{1}{2}\left(H(x_{k+1}) + H(x_k)\right)$$

$$= \sum_0^5 (x_k - c(f))\frac{1}{2}\left(H(x_{k-1}) - H(x_{k+1})\right)$$

$$= \sum_0^5 (x_k - c(f))\frac{1}{2}\left(H(x_{k-1}) + H(x_k)\right)$$

$$- \sum_0^5 (x_k - c(f))\frac{1}{2}\left(H(x_k) - H(x_{k+1})\right)$$

$$= - \sum_0^5 \left(\frac{x_k + x_{k+1}}{2} - c(f)\right)$$