2020 · 9 · 21

Thm. Bernoulli bond perco on $\mathbb{Z}^2$

- $p < p_c$: $\quad \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leqslant e^{-cn}$.

  $\exists\, c = c(p) > 0$,

- $p > p_c$: $\quad \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \xrightarrow{n\to\infty} \theta(p) > 0$.

$$\theta(p) \geqslant \frac{p - p_c}{p(1-p_c)} > 0 .$$

Remarks: ① $p > p_c$, $\quad \theta(p) \geqslant O(p - p_c)$.

Not optimal.

② $p < p_c$. exp. decay.

$p > p_c$. $\quad \theta(p) > 0$

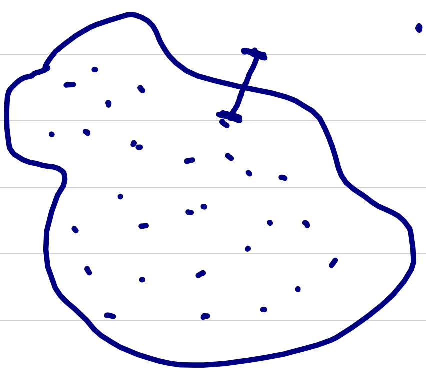
$p = p_c$? . $\quad \theta(p_c) = 0$. polynomial decay.

$\mathbb{P}[0 \leftrightarrow \partial\Lambda_n] = n^{-\frac{5}{48} + o(1)}$

Duminil-Copin - Tassion (2015).

- $S \subset V(\mathbb{Z}^2)$ a finite set containing 0.
- edge-boundary

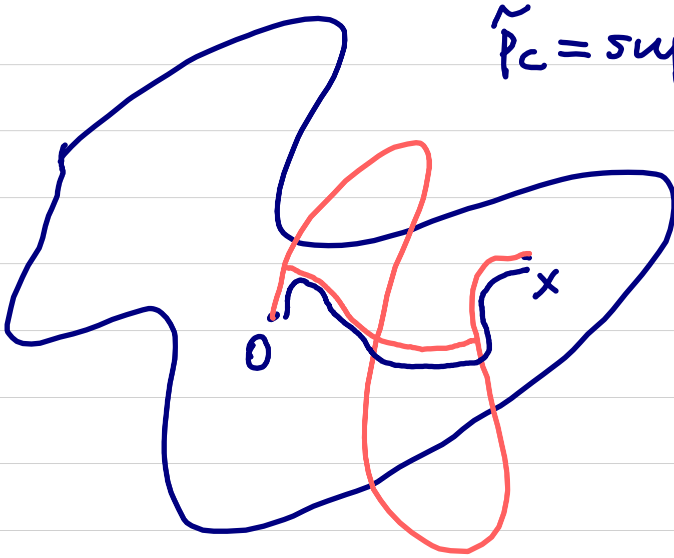
$$\Delta S := \{ \{x,y\} \in E : x \in S,\ y \notin S \}.$$



- $$\varphi_p(S) := p \sum_{(x,y) \in \Delta S} \mathbb{P}_p[0 \xleftrightarrow{S} x]$$

increasing in p.

$$\tilde{p}_c = \sup\{ p : \exists\, S \text{ s.t. } \varphi_p(S) < 1 \}$$



Goal: $p < \tilde{p}_c$, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$ ①

$p > \tilde{p}_c$, $\theta(p) \geq \dfrac{p - \tilde{p}_c}{p(1-\tilde{p}_c)}$. ②

consequence: $\tilde{p}_c = p_c$.

By ①, $p < \tilde{p}_c$, $\theta(p) = 0$. | $p < p_c$, $\theta(p) = 0$

By ②, $p > \tilde{p}_c$, $\theta(p) > 0$. | $p > p_c$, $\theta(p) = 0$.

Proof of ①:

$\tilde{p}_c = \sup\{p : \exists S \text{ with } \varphi_p(S) < 1\}$.

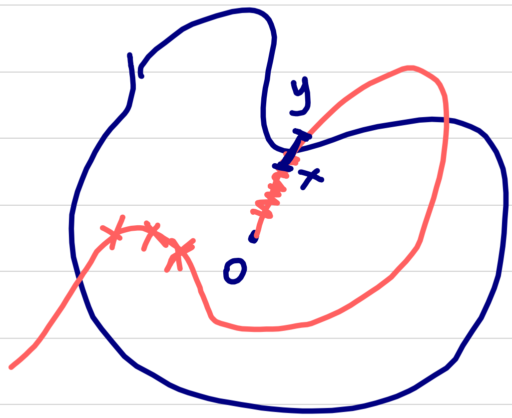
$p < \tilde{p}_c$, $\exists S$ with $\varphi_p(S) < 1$.

$S \subset \Lambda_{N-1}$.

$\underline{0 \leftrightarrow \partial\Lambda_{jN}}$. $\underline{0 \leftrightarrow \partial\Lambda_{(j-1)N}}$.

$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{jN}]$

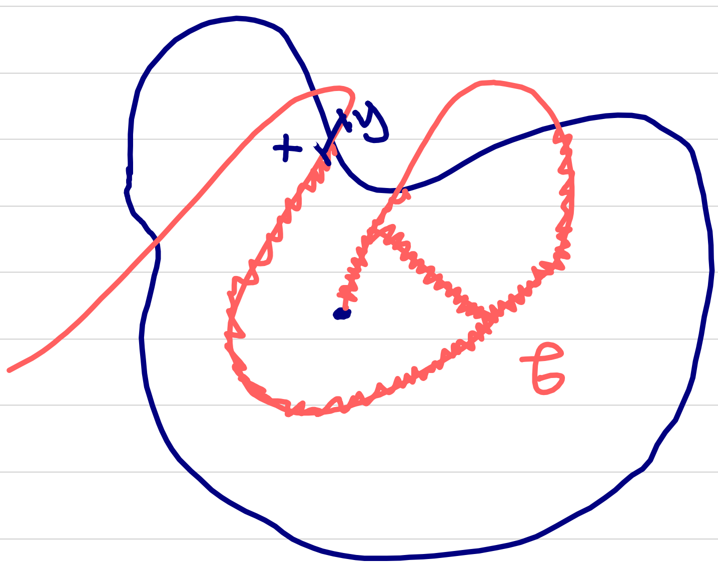
$\mathcal{C} = \{x \in S : 0 \xleftrightarrow{S} x\}$.



$0 \leftrightarrow \partial\Lambda_{jN}$.

$\exists\ (x,y) \in \Delta S$

$0 \xleftrightarrow{S} x$, $\{x,y\}$ open, $y \xleftrightarrow{\mathcal{C}^c} \partial\Lambda_{jN}$



$$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{jN}] \leq \sum_{(x,y)\in\Delta S} \mathbb{P}[0 \overset{S}{\leftrightarrow} x,\ (x,y) \text{ open},\ y \overset{\mathcal{C}^c}{\longleftrightarrow} \partial\Lambda_{jN}]$$

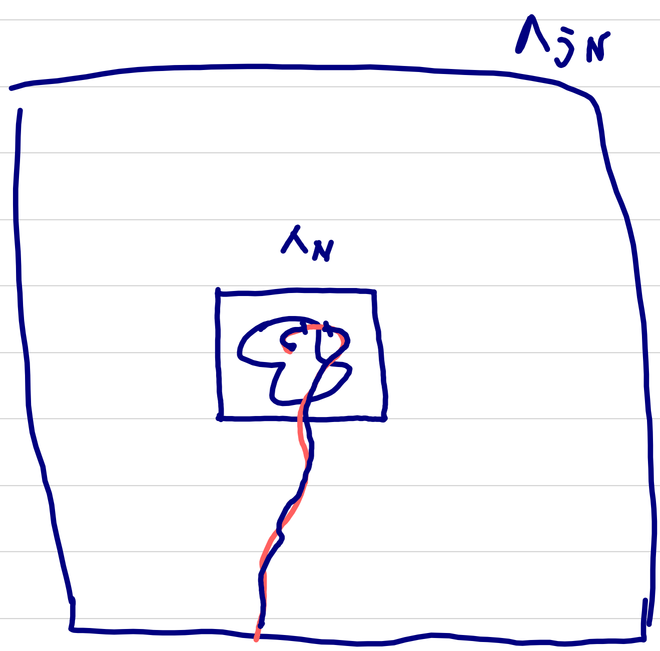
$$= \sum_{(x,y)\in\Delta S} \sum_{C\subset S} \mathbb{P}[\cdots,\ \mathcal{C}=C]$$

$$= \sum_{(x,y)\in\Delta S} \sum_{C\subset S} \mathbb{P}[0 \overset{S}{\leftrightarrow} x, \mathcal{C}=C,\ (x,y) \text{ open},\ y \overset{C^c}{\longleftrightarrow} \partial\Lambda_{jN}]$$

$$= \sum_{(x,y)\in\Delta S} \sum_{C\subset S} \mathbb{P}[0 \overset{S}{\leftrightarrow} x, \mathcal{C}=C] \cdot p \cdot \mathbb{P}[y \overset{C^c}{\longleftrightarrow} \partial\Lambda_{jN}]$$

$$\mathbb{P}[y \overset{C^c}{\longleftrightarrow} \partial\Lambda_{jN}]$$

$$\leq \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(j-1)N}]$$



$$\leq \sum_{(x,y)\in\Delta S} \sum_{C\subset S} \mathbb{P}[0 \overset{S}{\leftrightarrow} x, \mathcal{C}=C] \cdot p \cdot \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(j-1)N}]$$

$$= \underbrace{\sum_{(x,y)\in\Delta S} \mathbb{P}[0 \xleftrightarrow{S} x] \cdot p} \cdot \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(j-1)N}]$$

$$= \underline{\varphi_p(S) \cdot \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(j-1)N}]}.$$

$$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{jN}] \leq \varphi_p(S) \cdot \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(j-1)N}].$$

$$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{jN}] \leq \varphi_p(S)^{j}.$$

$$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{n}] \leq e^{-cn}.$$

Lemma.

$$\frac{d}{dp}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \geq \frac{1}{p(1-p)}\underline{\left(\inf_{S\subset\Lambda_n;\, 0\in S}\varphi_p(S)\right)}\left(1-\mathbb{P}_p[0\leftrightarrow\partial\Lambda_n]\right)$$

$p > \tilde{p}_c$, $\quad \tilde{p}_c = \sup\{p : \exists\, S \text{ with } \varphi_p(S) < 1\}$

$$\varphi_p(S) \geq 1, \quad \forall s.$$

$$\underbrace{\frac{d}{dp}\underline{\mathbb{P}_p[0\leftrightarrow\partial\Lambda_n]}}_{f(p)} \geq \frac{1}{p(1-p)}\left(1-\underline{\mathbb{P}_p[0\leftrightarrow\partial\Lambda_n]}\right)$$

$$f'(p) \geq \frac{1}{p(1-p)}(1-f(p)).$$

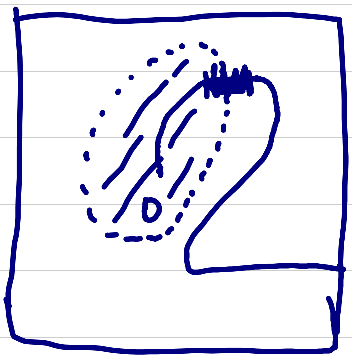
$$\frac{f'(p)}{1-f(p)} \geq \frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}$$

integrate from $\tilde{p}_c$ to p,

$$\log(1-f(\tilde{p}_c)) - \log(1-f(p)) \geq \log\frac{p(1-\tilde{p}_c)}{(1-p)\tilde{p}_c}$$

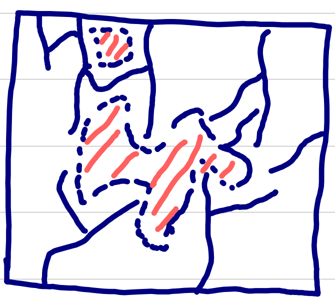
$$\frac{1-f(\tilde{p}_c)}{1-f(p)} \geq \frac{p(1-\tilde{p}_c)}{(1-p)\tilde{p}_c} \longrightarrow f(p) \geq \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}.$$

$$\frac{d}{dp}\mathbb{P}_p[0\leftrightarrow\partial\Lambda_n]=\sum_{e\in E(\Lambda_n)}\mathbb{P}_p[e \text{ pivotal for } \{0\leftrightarrow\partial\Lambda_n\}]$$



$$=\sum_{e\in E(\Lambda_n)}\frac{1}{1-p}\mathbb{P}_p\left[\begin{matrix}e \text{ pivotal for } \{0\leftrightarrow\partial\Lambda_n\}\\ 0\not\leftrightarrow\partial\Lambda_n\end{matrix}\right]$$

$$\mathscr{C}=\{x\in\Lambda_n:\ x\not\leftrightarrow\partial\Lambda_n\}$$



$\forall$ $S\subset\Lambda_n$ such that $0\in S$,

$\{\mathscr{C}=S\}$

$e=(x,y)$ is pivotal for $\{0\leftrightarrow\partial\Lambda_n\}$

and $\{0\not\leftrightarrow\partial\Lambda_n\}$

$$0\overset{S}{\leftrightarrow}x,\ e=(x,y)\in\Delta S$$

$$\mathbb{P}[e \text{ pivotal for } \{0\leftrightarrow\partial\Lambda_n\},\ \{0\not\leftrightarrow\partial\Lambda_n\}]$$

$$=\sum_{\substack{S\subset\Lambda_n\\0\in S}}\sum_{(x,y)\in\Delta S}\mathbb{P}_p[0\overset{S}{\leftrightarrow}x,\ \mathscr{C}=S].$$

$$\frac{d}{dp}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\ \sum_{(x,y)\in\Delta S} \mathbb{P}[\underline{0 \overset{S}{\leftrightarrow} x}, \underline{\mathcal{C}=S}]$$

$$= \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\ \underbrace{\sum_{(x,y)\in\Delta S} \mathbb{P}[0 \overset{S}{\leftrightarrow} x]}_{}\cdot\, \mathbb{P}[\mathcal{C}=S]$$

$$= \frac{1}{p(1-p)}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \underline{\varphi_p(S)}\ \mathbb{P}[\mathcal{C}=S]$$

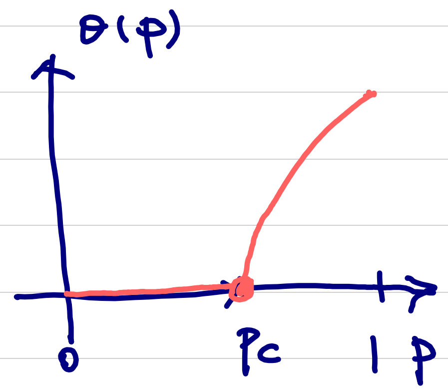
$$\geq \frac{1}{p(1-p)}\left(\inf_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S)\right)\underbrace{\sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \mathbb{P}[\mathcal{C}=S]}_{\mathbb{P}[0 \not\leftrightarrow \partial\Lambda_n]}$$

$$\frac{d}{dp}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$$

$$\geq \frac{1}{p(1-p)}\left(\inf_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S)\right)\left(1-\mathbb{P}_p[0\leftrightarrow\partial\Lambda_n]\right).$$

$$\theta(p) \geq \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}, \quad \forall\, p > \tilde{p}_c.$$

Thm. $p_c = \frac{1}{2}$. $\theta(p_c) = 0$.



$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty]$$

Lemma. $\theta(\frac{1}{2}) = 0$.

Pf: ω bond percolation on $\mathbb{Z}^2$

$\omega^*(e^*) = 1 - \omega(e)$ on $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$

$\omega \sim \mathbb{P}_{\underline{p}}$ on $\mathbb{Z}^2$, $\omega^* \sim \mathbb{P}_{\underline{1-p}}$ on $(\mathbb{Z}^2)^*$

$p = 1 - p$, $p = \frac{1}{2}$. $\omega \sim \mathbb{P}_{\frac{1}{2}}$ on $\mathbb{Z}^2$

$\omega^* \sim \mathbb{P}_{\frac{1}{2}}$ on $(\mathbb{Z}^2)^*$

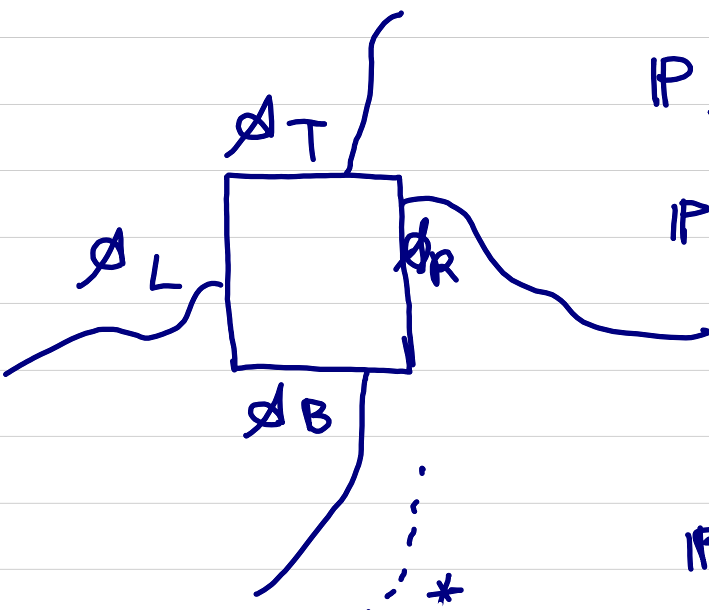
prove by induction. $\theta(\frac{1}{2}) > 0$.

$\mathbb{P}[\exists\ \infty\text{-cluster}] = 1$.

$\mathbb{P}[\exists\ \infty\text{-cluster}] = 1,$

$\mathbb{P}[\Lambda_n \leftrightarrow \infty] \to 1.$

$\forall \varepsilon > 0,\quad \exists\, n,\quad \mathbb{P}[\Lambda_n \leftrightarrow \infty] \geq 1 - \varepsilon^4.$



$$\mathbb{P}[\mathcal{A}_L^c \cap \mathcal{A}_R^c \cap \mathcal{A}_T^c \cap \mathcal{A}_B^c] \leq \varepsilon^4$$

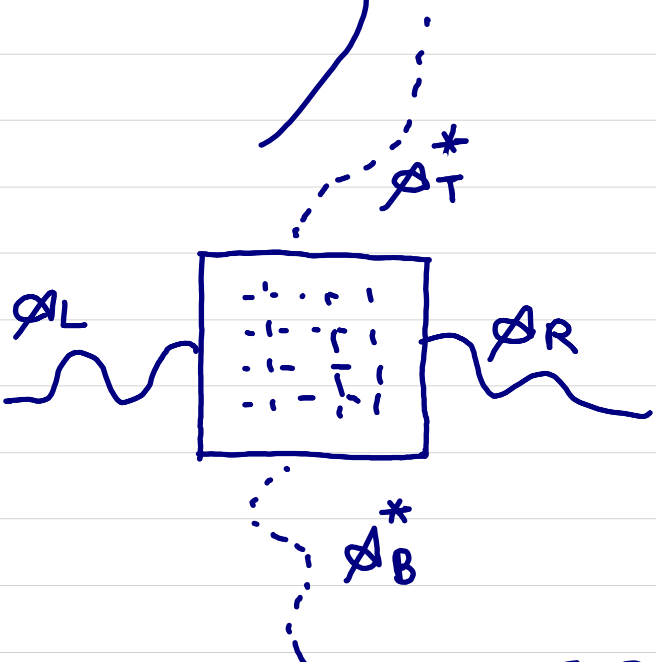
$$\geq$$

$$\mathbb{P}[\mathcal{A}_L^c]\,\mathbb{P}[\mathcal{A}_R^c]\,\mathbb{P}[\mathcal{A}_T^c]\,\mathbb{P}[\mathcal{A}_B^c]$$

$$=$$

$$\mathbb{P}[\mathcal{A}_L^c]^4$$

$$\mathbb{P}[\mathcal{A}_L^c] \leq \varepsilon$$



$$\mathbb{P}[\mathcal{A}_L \cap \mathcal{A}_R \cap \mathcal{A}_T^* \cap \mathcal{A}_B^*]$$

$$\geq 1 - 4\varepsilon.$$

$\xi_n = \{$all edges in Λ_n are closed$\}$

$$\mathbb{P}[\exists\ 2\ \infty\text{-clusters}] \leq \mathbb{P}[\xi_n \cap \mathcal{A}_L \cap \mathcal{A}_R \cap \mathcal{A}_T^* \cap \mathcal{A}_B^*] > 0$$

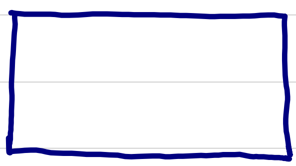
contradiction!

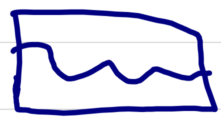
Lemma. $\theta(\frac{1}{2})=0$

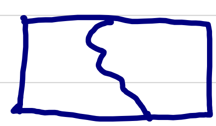
Proof of Thm.: $\begin{cases} \theta(p)=0, & p<p_c \\ \theta(p)>0, & p>p_c. \end{cases}$

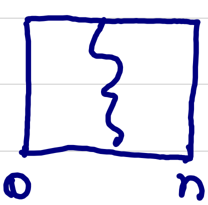
$p_c \geq \frac{1}{2}$. Goal: $p_c \leq \frac{1}{2}$.

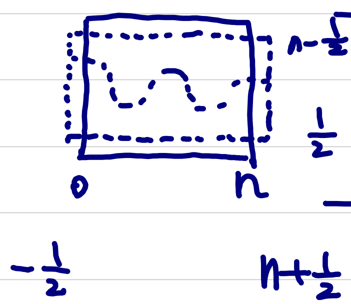
$p=\frac{1}{2}$. RSW. estimate

R: 

$C_h(R) :=$ 

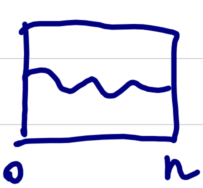
$C_v(R) :=$ 

$\mathbb{P}\Big[$ 

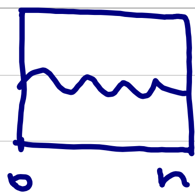
$\Big] + \mathbb{P}\Big[$ 

$\Big] = 1$

$\wedge|$

$\mathbb{P}\Big[$ 

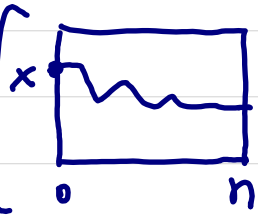
$\Big]$

$P\Big[$  $\Big] \geq \frac{1}{2}$. RSW estimate.

Goal: $p_c \geq \frac{1}{2}$.

prove by induction. $p_c > \frac{1}{2}$.

exp. decay: $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$.

$\frac{1}{2} \leq \mathbb{P}\Big[$  $\Big] \leq \sum_{x \in \partial_L \Lambda_n} \mathbb{P}[x \leftrightarrow x + \partial\Lambda_n]$

$\leq n e^{-cn}$

$\frac{1}{2} \leq n \cdot e^{-cn}$, $\forall n$

contradiction.

conclusion: $p_c = \frac{1}{2}$.