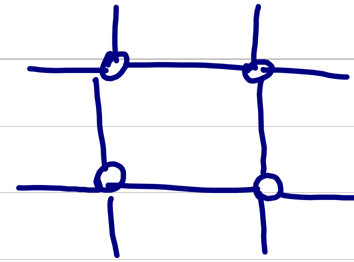
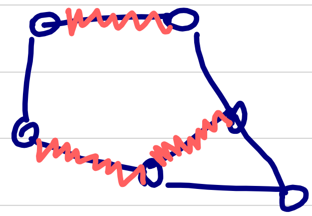
$G=(V,E)$.



configuration $\omega \in \{0,1\}^E$



$\mathbb{P}_p[\omega(e_1)=1,\cdots,\omega(e_n)=1]=p^n$,

$\forall\, e_1,\cdots,e_n \in E$.

partial order on $\{0,1\}^E$

$\omega \le \omega'$ iff $\omega(e)\le\omega'(e)$, $\forall e\in E$.

Def. An event A is increasing if $\mathbb{1}_A$ is increasing.

$\omega\in A$, $\omega'\ge\omega \Rightarrow \omega'\in A$.

Example.

A

increasing.

Lemma. [Monotonicity] $p \le p'$

A : increasing event. $\mathbb{P}_p[A] \le \mathbb{P}_{p'}[A]$.

Pf: (grande coupling).

$\{U_e, e \in E\}$ iid. unif$[0,1]$.

$$\omega_p(e) = \mathbb{1}\{U_e \le p\}$$

claim: $\omega_p \sim \mathbb{P}_p$.

$\{\omega_p(e), e \in E\}$ indept.

$$\mathbb{P}[\omega_p(e) = 1] = \mathbb{P}[U_e \le p] = p.$$

$p \le p'$, $\underline{\omega_p \le \omega_{p'}}$.

$$\omega_p(e) = \mathbb{1}\{U_e \le p\}$$

$$\le \mathbb{1}\{U_e \le p'\} = \omega_{p'}(e).$$

$$\mathbb{P}_p[A] = \mathbb{P}[\omega_p \in A] \le \mathbb{P}[\omega_{p'} \in A] = \mathbb{P}_{p'}[A].$$

Lemma. [FKG inequality]. f, g increasing.

$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\,\mathbb{E}_p[g].$$

$f = \mathbb{1}_A$, $g = \mathbb{1}_B$, A, B increasing.

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B].$$

$$\mathbb{P}_p[A \mid B] \geq \mathbb{P}_p[A].$$

Pf: $E = \{e_1, \cdots, e_N\}$. prove by induction on N.

$N=1$. $LHS = \mathbb{E}[fg] = f(1)g(1) \cdot p + f(0)g(0)(1-p)$.

$RHS = (f(1)p + f(0)(1-p))(g(1)p + g(0)(1-p))$.

$LHS - RHS = p(1-p)\left(f(1)g(1) + f(0)g(0) - f(0)g(1) - f(1)g(0)\right)$

$$= p(1-p)\underbrace{(f(1)-f(0))}_{\geq 0}\underbrace{(g(1)-g(0))}_{\geq 0} \geq 0.$$

Assume $N = n$. $\quad N = n+1$.

$$\text{LHS} = \mathbb{E}[fg]$$

$$= \mathbb{E}[f(w(e_1), \cdots, \underline{w(e_{n+1})})\, g(w(e_1), \cdots, \underline{w(e_{n+1})})]$$

$$= \mathbb{E}\big[\mathbb{E}[f(\cdots)\, g(\cdots) \mid w(e_{n+1})]\big]$$

$$= p\, \mathbb{E}[f(\underline{w(e_1), \cdots, w(e_n)}, 1)\, g(\underline{\cdots}, 1)]$$

$$+ (1-p)\, \mathbb{E}[f(\cdots, 0)\, g(\cdots, 0)]$$

$$\underset{\text{ind. hypo.}}{\geq} p\, \mathbb{E}[f(\cdots, 1)]\, \mathbb{E}[g(\cdots, 1)]$$

$$+ (1-p)\, \mathbb{E}[f(\cdots, 0)]\, \mathbb{E}[g(\cdots, 0)]$$

$$\underset{N=1.}{\geq} \mathbb{E}[f] \cdot \mathbb{E}[g].$$

finitely many edges

$\downarrow$ approximation + cvg thm.

infinitely many edges

pivotal.

For a configuration ω, fix $e \in E$,

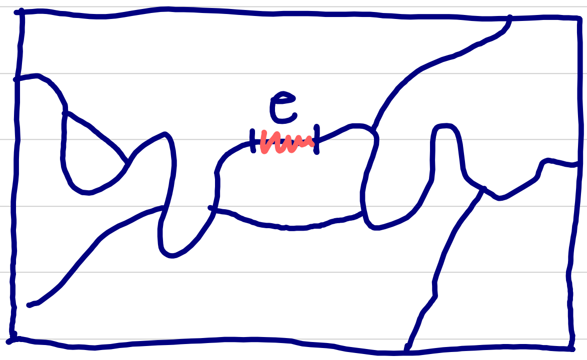
$$\omega^e(f) = \begin{cases} \omega(f), & f \neq e \\ 1, & f = e. \end{cases}$$

$$\omega_e(f) = \begin{cases} \omega(f), & f \neq e \\ 0, & f = e. \end{cases}$$

e is pivotal for A

$$\{\omega : \omega^e \in A, \ \omega_e \notin A\}.$$

Example:



Lemma. [Russo's formula].

A increasing. finitely many edges.

$$\frac{d}{dp} \mathbb{P}_p[A] = \sum_{e \in E} \mathbb{P}_p[\, e \text{ is pivotal for } A]$$

Pf: $\{e_1, \dots, e_N\}$. $\vec{p} = (p_1, \dots, p_N)$

$$\mathbb{P}[\omega(e_j) = 1] = p_j, \quad 1 \le j \le N$$

$$\frac{\partial}{\partial p_j} \mathbb{P}_{\vec{p}}[A] = \frac{1}{\varepsilon}\left(\underline{\mathbb{P}_{\vec{p}+\varepsilon \vec{e}_j}}[A] - \underline{\mathbb{P}_{\vec{p}}}[A]\right)_\Delta$$

$\mathbb{P}_{\vec{p}}$: grande coupling. $\omega_{\vec{p}}(e_i) = \mathbb{1}\{U(e_i) \le p_i\}$.

$\omega_{\vec{p}} \sim \mathbb{P}_{\vec{p}}$.

$\omega = \omega_{\vec{p}}, \quad \omega' = \omega_{\vec{p}+\varepsilon \vec{e}_j}$

$\omega(f) = \omega'(f), \quad f \neq e_j$.

$\omega(e_j) = \mathbb{1}\{\underline{U(e_j)} \le p_j\}, \quad \omega'(e_j) = \mathbb{1}\{U(e_j) \le p_j + \varepsilon\}$.

$$\frac{\partial}{\partial p_j} \mathbb{P}_{\vec{p}}[A] = \lim \frac{1}{\varepsilon}\left(\mathbb{P}_{\vec{p}+\varepsilon\vec{e_j}}[A] - \mathbb{P}_{\vec{p}}[A]\right)$$

$$= \lim \frac{1}{\varepsilon}\left(\mathbb{P}[\omega' \in A] - \mathbb{P}[\omega \in A]\right)$$

$$= \lim \frac{1}{\varepsilon} \mathbb{P}[\omega' \in A, \ \omega \notin A]$$

$$= \lim \frac{1}{\varepsilon} \cdot \mathbb{P}[\underline{U(e_j) \in (p_j, p_j+\varepsilon)}, \ \underline{\omega^{e_j} \in A, \ \omega_{e_j} \notin A}]$$

$$= \mathbb{P}[\ e_j \text{ is pivotal for } A].$$

$$\frac{d}{dp} \mathbb{P}_p [A]$$

$$= \sum_j \frac{\partial}{\partial p_j} \mathbb{P}_{\vec{p}}[A]$$

$$= \sum_j \mathbb{P}[\ e_j \text{ is pivotal for } A]$$

$$= \text{RHS}.$$