Conformal Invariance in 2D Lattice Models

Part 1: Bernoulli Percolation

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Part 1: Bernoulli Percolation
Part 2: Random Cluster Model
Part 3: Ising Model

Goal

Bernoulli bond percolation

- Monotonicity
- FKG inequality
- Russo's formula

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- critical value
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Bernoulli Bond Percolation

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- $G = (V, E)$ is a graph.
- configuration $\omega \in \{0, 1\}^E$.
- each configuration ω can also be viewed as a subgraph of G.

The Bernoulli percolation measure of parameter $p \in [0, 1]$:

$$\mathbb{P}_p[\omega(e_1) = 1, \cdots, \omega(e_n) = 1] = p^n, \quad \forall e_1, \dots, e_n \in E, \forall n \geq 1.$$

- The parameter $p \in [0, 1]$ is called edge-weight.
- The σ-algebra of measurable events is the smallest σ-algebra containing events depending on finitely many edges.

Monotonicity

Consider the partial order on $\{0,1\}^E$ given by

$$\omega \leq \omega' \quad \text{if and only if} \quad \omega(e) \leq \omega'(e), \text{ for all } e \in \mathbb{E}.$$

- An event A is said to be increasing if 1_A is increasing.
- An event A is increasing if

$$\omega \in A \text{ and } \omega' \geq \omega, \quad \text{implies } \omega' \in A.$$

Lemma (Monotonicity)

Let $p \leq p'$. Then for any increasing event A, we have

$$\mathbb{P}_p[A] \leq \mathbb{P}_{p'}[A].$$

FKG Inequality

Lemma (FKG Inequality)

Let $p \in [0,1]$, for any two increasing functions f and g, we have

$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\mathbb{E}_p[g].$$

In particular, for any two increasing events A and B, we have

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\mathbb{P}_p[B].$$

Russo's Formula

For a configuration ω,

- let ω^e be equal to ω except at e and $\omega^e(e) = 1$
- let ω_e be equal to ω except at e and $\omega_e(e) = 0$

Let A be an increasing event, we say that e is pivotal for A if

$$\omega^e \in A, \quad \omega_e \notin A.$$

- The event $\{e \text{ is pivotal for } A\}$ does not depend on the status of e.

Lemma (Russo's formula)

Let A be an increasing event depending on the states of a finite set of edges E. Then

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_{e \in E} \mathbb{P}_p[e \text{ is pivotal for } A].$$

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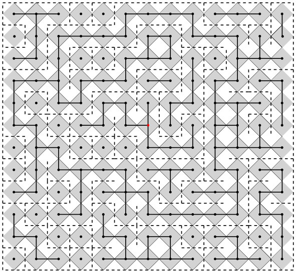

Question

We are interested in the connectivity property when $G = \mathbb{Z}^2$:

$$\mathbb{P}_p[0 \leftrightarrow \infty] > 0?$$

Homework : check that the event $\{0 \leftrightarrow \infty\}$ is in the σ-algebra.

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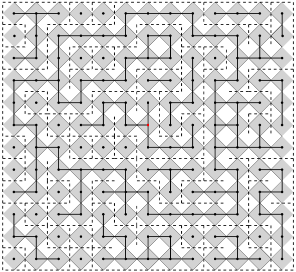

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Theorem

There exists $p_c \in (0,1)$ *such that*

$$\mathbb{P}_p[0 \leftrightarrow \infty] = 0 \text{ for } p < p_c; \quad \mathbb{P}_p[0 \leftrightarrow \infty] > 0 \text{ for } p > p_c.$$

Proof

Existence of p_c ; $p_c > 0$; $p_c < 1$ (duality).

Duality

$G = (V, E)$ is a plane graph, the dual graph of G is a graph $G^* = (V^*, E^*)$:

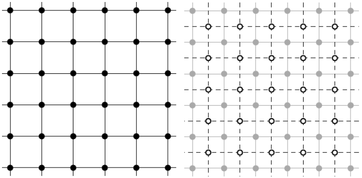
- it has a vertex for each face of G
- it has an edge whenever two faces of G are separated from each other by an edge.

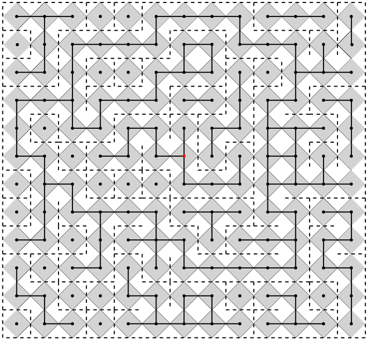
The dual graph of $\mathbb{Z}^2$ is $\mathbb{Z}^2 + (1/2, 1/2)$.

Given $\omega \in \{0, 1\}^E$, define

$$\omega^*(e^*) = 1 - \omega(e), \forall e \in E.$$

- If $\omega \sim \mathbb{P}_p$ on G, then $\omega^* \sim \mathbb{P}_{1-p}$ on G^*.





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Uniqueness of infinite cluster

Theorem

For Bernoulli bond percolation on $\mathbb{Z}^2$, either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.

- $\mathbb{P}_p[\exists$ infinite cluster$] = 0$ when $\theta(p) = 0$.
- $\mathbb{P}_p[\exists$ infinite cluster$] = 1$ when $\theta(p) > 0$: Ergodicity
- When $\theta(p) > 0$, there exists a unique infinite cluster.

Ergodicity

- Let $\tau_x : \{0,1\}^{E(\mathbb{Z}^2)} \to \{0,1\}^{E(\mathbb{Z}^2)}$ be the shift by a vector $x \in \mathbb{Z}^2$ defined by

$$(\tau_x\omega)(\{a,b\}) = \omega(\{a+x, b+x\}), \quad \forall \{a,b\} \in E(\mathbb{Z}^2).$$

- For any event A, define $\tau_x A = \{\omega : \tau_x\omega \in A\}$.
- An event A is invariant under translation if $\tau_x A = A$ for any $x \in \mathbb{Z}^2$.

Lemma (Ergodicity of Bernoulli Percolation)

Bernoulli bond percolation on $\mathbb{Z}^2$ is ergodic, i.e. any event A which is invariant under translation satisfies $\mathbb{P}_p[A] \in \{0,1\}$.

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Subcritical : exponential decay

Theorem

Consider Bernoulli bond percolation on $\mathbb{Z}^2$.

- *If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

- *If $p > p_c$, then*

$$\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}.$$

Exponential decay

- Let S be a finite set of vertices containing 0.
- Its edge-boundary : $\Delta S = \{\{x,y\} \in E : x \in S, y \notin S\}$.
- For $p \in [0,1]$ and $0 \in S \subset \mathbb{Z}^2$, define

$$\varphi_p(S) := p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p \left[0 \overset{S}{\longleftrightarrow} x \right].$$

Set

$$\tilde{p}_c = \sup\{p \in [0,1] : \exists S \text{ with } \varphi_p(S) < 1\}.$$

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The critical value

Theorem

For Bernoulli bond percolation on $\mathbb{Z}^2$, we have

$$p_c = 1/2, \quad \theta(p_c) = 0.$$

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- critical value open
- continuity of PT ✓ $d \geq 11$; conjecture for others

Bernoulli site percolation

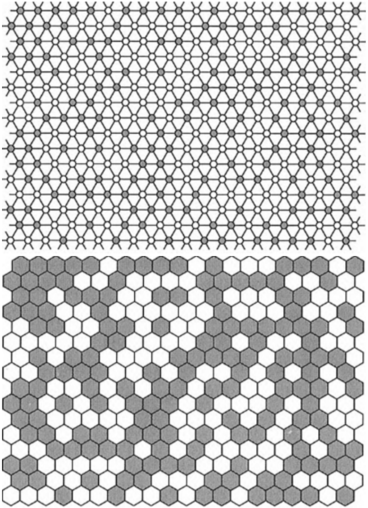
- $G = (V, E)$ is a graph
- configuration $\omega \in \{\bullet, \circ\}^V$

The Bernoulli site percolation on G is a probability measure $\mathbb{P}_p$ on ω :

$$\mathbb{P}_p[\omega(x_1) = \bullet, \cdots, \omega(x_n) = \bullet] = p^n,$$

for all $x_1, \ldots, x_n \in V$, and for all $n \geq 1$.

- triangular lattice $\mathbb{T} = (V(\mathbb{T}), E(\mathbb{T}))$



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Critical Bernoulli site percolation on $\mathbb{T}$

- G : honeycomb
- G^* : triangular
- G_δ,G^*_δ :scaled by δ
- $(\Omega; A, B, C, D)$: a quad
- $(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta)$: approximation on G_δ
- $\mathcal{C}_\delta(\Omega; A, B, C, D)$: $\exists$ a black crossing in Ω^*_δ

Theorem (Smirnov 2001)

- *The probability of $\mathcal{C}_\delta$ is convergent :*

$$\mathbb{P}[\mathcal{C}_\delta(\Omega; A, B, C, D)] \to f(\Omega; A, B, C, D), \quad \text{as } \delta \to 0.$$

- *The function f is conf. inv. : for any conformal map ϕ on Ω,*

$$f(\phi(\Omega); \phi(A), \phi(B), \phi(C), \phi(D)) = f(\Omega; A, B, C, D).$$

- *When Ω equals the equilateral triangle with three vertices A, B, C,*

$$f(\Omega; A, B, C, D) = |CD|/|CA|.$$

Conformal Invariance : the strategy

- For $z \in \Omega$,

$$E_A^\delta(z) = \{\exists \text{ a black path } A_\delta, z_\delta \mid B_\delta, C_\delta\}, \quad H_A^\delta(z) = \mathbb{P}[E_A^\delta(z)]$$

- $E_B^\delta(z), E_C^\delta(z)\,;\, H_B^\delta(z) = \mathbb{P}[E_B^\delta(z)], H_C^\delta(z) = \mathbb{P}[E_C^\delta(z)]$.
- Define

$$H^\delta(z) = H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z), \quad S^\delta(z) = H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z).$$

The strategy

- **Tightness.** The family $\{H_A^\delta, H_B^\delta, H_C^\delta\}_{\delta>0}$ is tight under the topology of uniform convergence.
- **Holomorphicity.** Any subsequential limit H, S are holomorphic.
- **Boundary Value.** $S \equiv 1$ and $H_A(z) = (2\Re H(z) + 1)/3$ where H is the conformal map from Ω onto $\triangleright$ which sends (A, B, C) to $(1, \tau, \tau^2)$.

Conformal Invariance : Color switching

A function $f : \Omega \to \mathbb{C}$

- It is holomorphic if, for any $z \in \Omega$, the derivative $f'(z) = \lim_{|\epsilon| \to 0}(f(z+\epsilon) - f(z))/\epsilon$ exists.
- It is holomorphic if, for any simple closed smooth curve γ, we have $\oint_\gamma f = 0$. (Morera)

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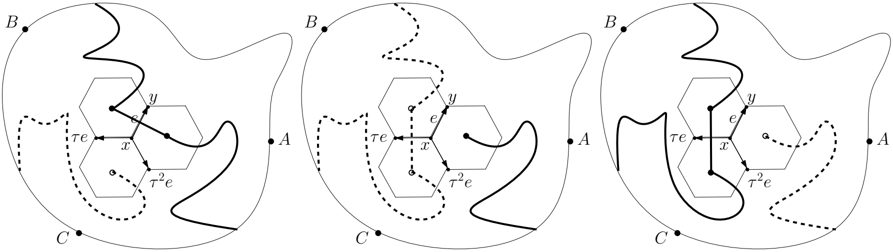

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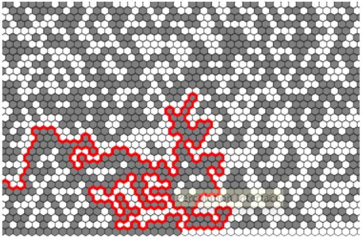

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- *The interface converges to SLE(6) (Schramm Loewner Evolution).*

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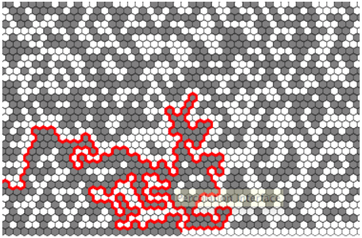

Decay of the density

- When $p < p_c$
- $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c(p)n}$.

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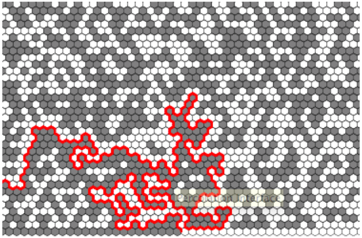

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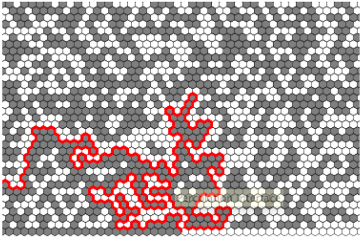

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- $\mathbb{P}_{p_c}[0 \leftrightarrow \partial\Lambda_n] = n^{-5/48+o(1)}$.
- $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \to \theta(p) > 0$.

Moreover,

$$\theta(p) = (p - p_c)^{5/36+o(1)}, \quad \text{as } p \downarrow p_c.$$