Conformal Invariance in 2D Lattice Models

Part 0: Introduction

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Ising Model

Curie temperature [Pierre Curie, 1895]
Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.
Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$
\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))
$$

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$
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- $\beta \approx \beta_c$ : critical
- $\beta < \beta_c$ : chaotic
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**Question**

\[ \beta_c = ? \]

**Answer** [Kramers-Wannier, Onsager-Kaufman, 1940]

Ising model on \( \mathbb{Z}^2 \):

\[ \beta_c = \frac{1}{2} \log(1 + \sqrt{2}). \]
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Conformally invariant (CI).
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**Question**

Critical phase ?

**Answer**

Conformally invariant (CI). What does it mean?
Ising Model

\( T \gg T_c \) \quad \text{T} \sim T_c \quad \text{T} \ll T_c \)

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What does it mean?

Correlation function
\( \mu[\sigma_{z_1} \cdots \sigma_{z_n}] \rightarrow \phi(z_1, \ldots, z_n) \).

Schramm Loewner Evolution (SLE)
The law of interfaces is CI.
Conformal Invariance of Interfaces
Conformal Invariance of Interfaces

SLE [O. Schramm 1999]

A random fractal curve: conformal invariance, domain Markov property

Classification: SLE (κ), κ > 0.
Conformal Invariance of Interfaces
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SLE[O. Schramm 1999]
A random fractal curve:
- conformal invariance
- domain Markov property
Classification: SLE($\kappa$), $\kappa > 0$. 

A way to construct random conformally invariant fractal curves, introduced in 1999 by Oded Schramm (1961-2008)

Percolation $\rightarrow$ SLE(6)
Uniform Spanning Tree $\rightarrow$ SLE(8)
Conformal Invariance in Ising Model

[Chelkak-Smirnov, Invent.Math. ’10]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE(3).
Tentative syllabus

- Bernoulli percolation (6 lectures)
- Random cluster model (5 lectures)
- Ising model (3 lectures)

References:

- W. Werner. Random planar curves and Schramm-Loewner evolutions.
- W. Werner. Lectures on two-dimensional critical percolation.