Conformal Invariance in 2D Lattice Models

Part 0: Introduction

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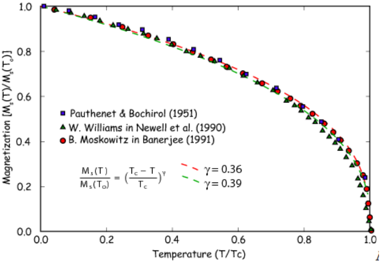
Ising Model

Pierre and Marie Skłodowska-Curie, 1895

Curie temperature [Pierre Curie, 1895]

Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.



Ising Model

Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

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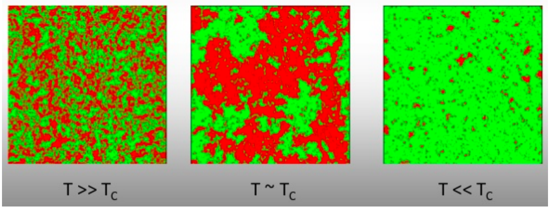

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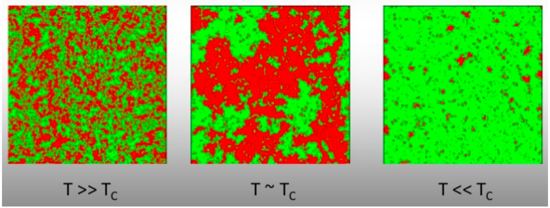

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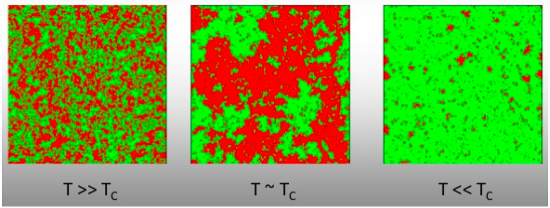

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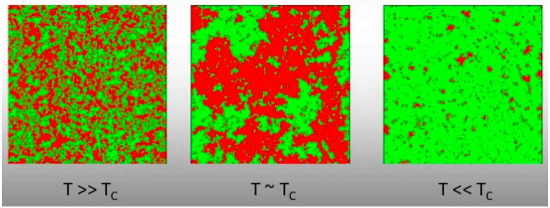

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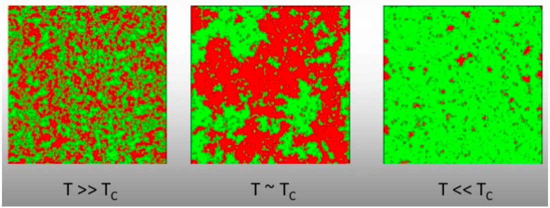

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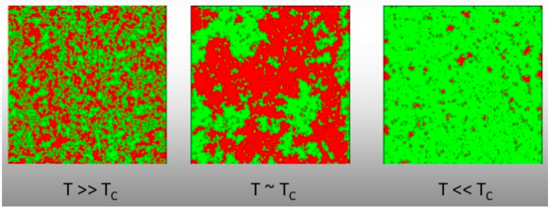

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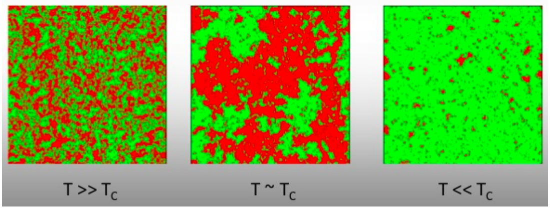

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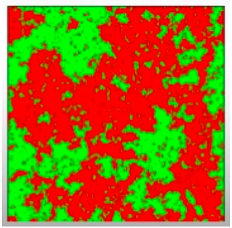
Correlation function

$\mu[\sigma_{z_1}\cdots\sigma_{z_n}] \to \phi(z_1,\ldots,z_n)$.

Schramm Loewner Evolution (SLE)

The law of interfaces is CI.

Conformal Invariance of Interfaces



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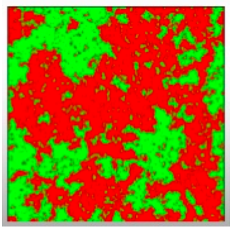

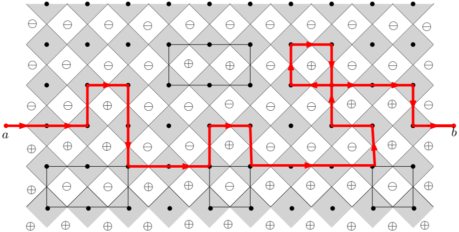

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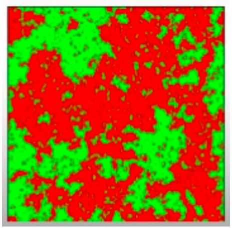

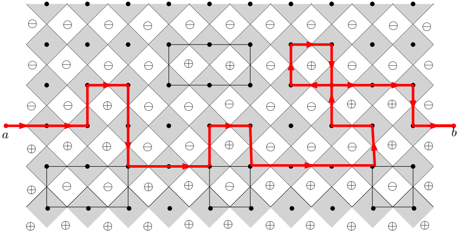

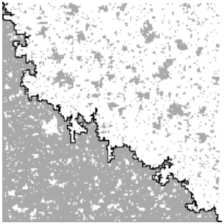

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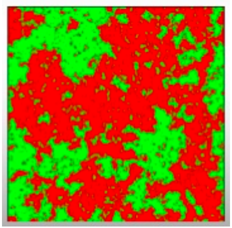
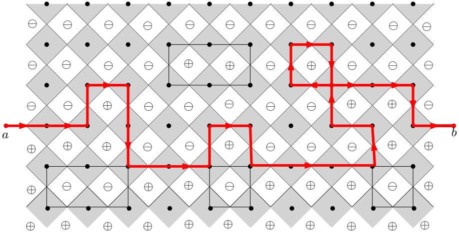
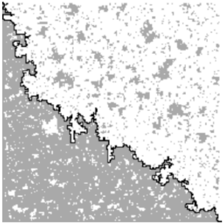






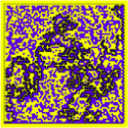
SLE[O. Schramm 1999]

A random fractal curve :

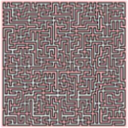
- conformal invariance
- domain Markov property

Classification : SLE(κ), $\kappa > 0$.

A way to construct **random conformally invariant fractal curves,** introduced in 1999 by **Oded Schramm** (1961-2008)



Percolation→SLE(6)

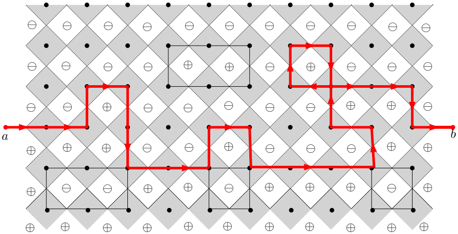


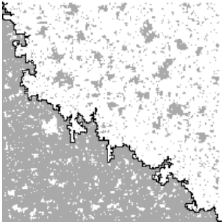
Uniform Spanning Tree →SLE(8)

Conformal Invariance in Ising Model

[Chelkak-Smirnov, Invent.Math. ’10]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE(3).





Tentative syllabus

- Bernoulli percolation (6 lectures)
- Random cluster model (5 lectures)
- Ising model (3 lectures)

References :

- W. Werner.
  Random planar curves and Schramm-Loewner evolutions.
- W. Werner.
  Lectures on two-dimensional critical percolation.
- H. Duminil-Copin and S. Smirnov.
  Conformal invariance of lattice models.