Classification problems in operator algebras and ergodic theory

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Operator algebras

We consider $*$-subalgebras $M \subset B(H)$, where the $*$-operation is the Hermitian adjoint.

- **Operator norm:**
  for $T \in B(H)$, we put $\|T\| = \sup\{\|T\xi\| \mid \xi \in H, \|\xi\| \leq 1\}$.

  C*-algebras: norm closed $*$-subalgebras of $B(H)$.

- **Weak topology:**
  $T_i \to T$ if and only if $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.

  Von Neumann algebras: weakly closed $*$-subalgebras of $B(H)$.

$\rightsquigarrow$ Intimate connections to group theory, dynamical systems, quantum information theory, representation theory, ...

Commutative operator algebras

- Unital commutative C*-algebras are of the form $C(X)$ where X is a compact Hausdorff space.

  $\rightsquigarrow$ algebraic topology, K-theory, continuous dynamics, geometric group theory

- Commutative von Neumann algebras are of the form $L^\infty(X, \mu)$ where (X, μ) is a standard probability space.

  $\rightsquigarrow$ ergodic theory, measurable dynamics, measurable group theory

Discrete groups and operator algebras

Let G be a countable (discrete) group.

- Left regular unitary representation $\lambda : G \to \mathcal{U}(\ell^2(G)) : \lambda_g \delta_h = \delta_{gh}$.
- $\text{span}\{\lambda_g \mid g \in G\}$ is the **group algebra** $\mathbb{C}[G]$.
- Take the norm closure: (reduced) **group C*-algebra** $C_r^*(G)$.
- Take the weak closure: **group von Neumann algebra** $L(G)$.

We have $G \subset \mathbb{C}[G] \subset C_r^*(G) \subset L(G)$.

At each inclusion, information gets lost $\rightsquigarrow$ natural rigidity questions.

Open problems

- Kaplansky's conjectures for torsion-free groups G.
  - Unit conjecture: the only invertibles in $\mathbb{C}[G]$ are multiples of group elements λ_g.
  - Idempotent conjecture: 0 and 1 are the only idempotents in $\mathbb{C}[G]$.
  - Kadison-Kaplansky: 0 and 1 are the only idempotents in $C_r^*(G)$.
- Kaplansky's direct finiteness conjecture: if k is a field and $a, b \in k[G]$ with $ab = 1$, then $ba = 1$. Holds if $\mathrm{char}\, k = 0$, using operator alg.
- Free group factor problem: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ if $n \neq m$?
- Connes' rigidity conjecture: $L(\mathrm{PSL}(n, \mathbb{Z})) \not\cong L(\mathrm{PSL}(m, \mathbb{Z}))$ if $3 \leq n < m$.
- Stronger form: if G has property (T) and $\pi : L(G) \to L(\Gamma)$ is a $*$-isomorphism, then $G \cong \Gamma$ and π is essentially given by such an isomorphism.

Operator algebras and group actions

Let G be a countable group.

Continuous dynamics and C*-algebras

An action $G \curvearrowright X$ of G by homeomorphisms of a compact Hausdorff space X gives rise to the C*-algebra $C(X) \rtimes_r G$.

Measurable dynamics and von Neumann algebras

An action $G \curvearrowright (X, \mu)$ of G by measure class preserving transformations of (X, μ) gives rise to a von Neumann algebra $L^\infty(X) \rtimes G$.

- These operator algebras contain $C(X)$, resp. $L^\infty(X)$, as subalgebras.
- They contain G as unitary elements $(u_g)_{g \in G}$.
- They encode the group action: $u_g \, F \, u_g^* = \alpha_g(F)$ where $(\alpha_g(F))(x) = F(g^{-1} \cdot x)$.

Amenable von Neumann algebras: full classification

Some run-up: Murray - von Neumann types.

Factor: a von Neumann algebra M with trivial center, i.e. $M \ncong M_1 \oplus M_2$.

A factor M is of

- type I if there are minimal projections, i.e. $M \cong B(H)$,
- type II_1 if not of type I and $1 \in M$ is a finite projection: if $v^*v = 1$, then $vv^* = 1$,
- type II_∞ if not of type II_1 but pMp of type II_1 for a projection $p \in M$,
- type III otherwise.

Theorem (Murray - von Neumann): every II_1 factor admits a faithful normal trace $\tau : M \to \mathbb{C}$. **Trace property:** $\tau(xy) = \tau(yx)$.

$\rightsquigarrow$ Type of $L^\infty(X) \rtimes G$ depends on the (non)existence of G-invariant measures on X, while $L(G)$ is always of type II_1.

The hyperfinite II_1 factor

Take $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \cdots$, where $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

$\rightsquigarrow$ Completion of direct limit: II_1 factor R.

Definition (Murray - von Neumann)

A von Neumann algebra M is called **approximately finite dimensional** (AFD) if there exists an increasing sequence of finite dimensional subalgebras $A_n \subset M$ with weakly dense union.

Theorem (Murray - von Neumann)

The II_1 factor R constructed above is the unique AFD factor of type II_1. It is called the hyperfinite II_1 factor.

What about other types? Which factors are AFD? $L^\infty(X) \rtimes G$?

Amenability

Definition (von Neumann, 1929)

A countable group G is amenable if there exists a finitely additive probability measure m on the subsets of G such that $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in G$ and $\mathcal{U} \subset G$.

$\rightsquigarrow$ Closely related to the Banach-Tarski paradox.

- (Banach - Tarski, 1924) It is possible to partition the ball of radius one into finitely many subsets,

  move these subsets by rotations and translations,

  and obtain two balls of radius one.

- Reason: group of motions of $\mathbb{R}^3$ is not amenable (as a discrete group).
- (Tarski, 1929) There is no paradoxical decomposition of the unit disk.
- Reason: group of motions of $\mathbb{R}^2$ is amenable (as a discrete group).

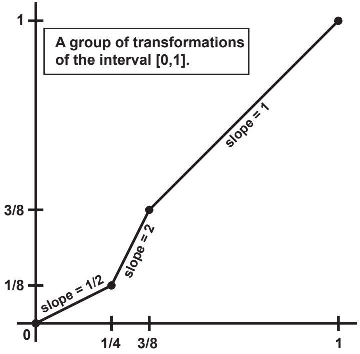
Examples

The following groups are **amenable.**

- Finite groups.
- Abelian groups.
- Stable under subgroups, direct limits and extensions.

The following groups are **non-amenable.**

- The free groups $\mathbb{F}_n$.
- Groups containing $\mathbb{F}_2$.
- Also other examples.



Open problem :

Is the Thompson group amenable ?

Amenability for von Neumann algebras

Definition (von Neumann, 1929)

A countable group G is amenable if there exists a finitely additive probability measure m on the subsets of G such that $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in G$ and $\mathcal{U} \subset G$.

$\rightsquigarrow$ Equivalently: there exists a G-invariant state $\omega : \ell^\infty(G) \to \mathbb{C}$.

Hakeda-Tomiyama: a von Neumann algebra $M \subset B(H)$ is amenable if there exists a conditional expectation $P : B(H) \to M$.

$\rightsquigarrow$ $L(G)$ and $L^\infty(X) \rtimes G$ are amenable whenever G is amenable.

Theorem (Connes, 1976)

Every amenable von Neumann algebra is AFD ! In particular, all amenable II_1 factors are isomorphic with R !

Modular theory: Tomita - Takesaki - Connes

Murray - von Neumann: II_1 factors admit a trace $\tau : M \to \mathbb{C}$, $\tau(xy) = \tau(yx)$.

Tomita - Takesaki: any faithful normal state $\omega : M \to \mathbb{C}$ on a von Neumann algebra M gives rise to a one-parameter group $\sigma_t^\omega \in \mathrm{Aut}(M)$ such that $\omega(xy) = \omega(y\,\sigma_{-i}^\omega(x))$ $\rightsquigarrow$ **KMS condition.**

Connes: this "time evolution" $(\sigma_t^\omega)_{t\in\mathbb{R}}$ is essentially independent of the choice of ω.

- Connes - Takesaki: every type III factor M is of the form $M \cong N \rtimes \mathbb{R}$ where N is of type II_∞.
- Restricting the action $\mathbb{R} \curvearrowright N$ to the center of N leads to an ergodic flow $\mathbb{R} \curvearrowright (Z, \eta)$.
- This is an isomorphism invariant of M.

Classification of amenable factors

Type III factor M $\rightsquigarrow$ ergodic flow $\mathbb{R} \curvearrowright (Z, \eta)$.

Definition (Connes)

A type III factor M is of

- type III_λ if the flow is periodic: $\mathbb{R} \curvearrowright \mathbb{R}/(\log \lambda)\mathbb{Z}$,
- type III_1 if the flow is trivial: $Z = \{\star\}$,
- type III_0 if the flow is properly ergodic.

Classification of amenable factors

- (Connes) For each of the following types, there is a unique amenable factor: type II_1, type II_∞, type III_λ with $0 < \lambda < 1$.
- (Connes, Krieger) The amenable factors of type III_0 are exactly classified by the associated flow.
- (Haagerup) There is a unique amenable III_1 factor.

Amenability for C*-algebras

The correct notion is: **nuclearity.**

The C*-algebra $C_r^*(G)$ is nuclear if and only if G is amenable.

Elliott program: classification of unital, simple, nuclear C*-algebras by K-theory and traces.

$\rightsquigarrow$ Huge efforts, by many people, over the past decades.

Currently approaching a final classification theorem,

for all unital, simple, nuclear C*-algebras satisfying a (needed) regularity property.

Beyond amenability: Popa's deformation/rigidity theory

Consider one of the most well studied group actions:

Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g \in G} (X_0, \mu_0) : (g \cdot x)_h = x_{g^{-1}h}$.

- $M = L^\infty(X) \rtimes G$ is a II_1 factor.
- Whenever G is amenable, we have $M \cong R$.

Superrigidity theorem (Popa, Ioana, V)

If G has property (T), e.g. $G = \mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$,

or if $G = G_1 \times G_2$ is a non-amenable direct product group,

then $L^\infty(X) \rtimes G$ remembers the group G and its action $G \curvearrowright (X, \mu)$.

More precisely: if $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes \Gamma$ for any other free, ergodic, probability measure preserving (pmp) group action $\Gamma \curvearrowright (Y, \eta)$,

then $G \cong \Gamma$ and the actions are conjugate (isomorphic).

Free groups

Theorem (Popa - V)

Whenever $n \neq m$, we have that $L^\infty(X) \rtimes \mathbb{F}_n \not\cong L^\infty(Y) \rtimes \mathbb{F}_m$,

for arbitrary free, ergodic, pmp actions of the free groups.

- If $L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$, there also exists an isomorphism π such that $\pi(L^\infty(X)) = L^\infty(Y)$.

  This is thanks to **uniqueness of the Cartan subalgebra.**

- Such a π induces an **orbit equivalence:** a measurable bijection $\Delta : X \to Y$ such that $\Delta(\mathbb{F}_n \cdot x) = \mathbb{F}_m \cdot \Delta(x)$ for a.e. $x \in X$.

- (Gaboriau) The L^2-Betti numbers of a group are invariant under orbit equivalence.

  We have $\beta_1^{(2)}(\mathbb{F}_n) = n - 1$.

L^2-Betti numbers of groups

- Let G be a countable group. View $\ell^2(G)$ as a left G-module (by left translation) and a right $L(G)$-module (by right translation).

- **Atiyah, Cheeger-Gromov, Lück:**
  define $\beta_n^{(2)}(G) = \dim_{L(G)} H^n(G, \ell^2(G))$.

- **Gaboriau:** invariant under orbit equivalence.

Conjecture (Popa, Ioana, Peterson)

If $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes \Gamma$ for some free, ergodic, pmp actions,
then $\beta_n^{(2)}(G) = \beta_n^{(2)}(\Gamma)$ for all $n \geq 0$.

Big dream (many authors)

Define some kind of L^2-Betti numbers for II_1 factors.
Prove that $\beta_1^{(2)}(L(\mathbb{F}_n)) = n - 1$.