Stochastic dualities for asymmetric interacting particle systems

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References: arxiv:1407.3367(to appear in PTRF), 1507.01478

Plan

1. (Self-)duality for SEP
2. KMP model
3. ASEP
4. A general construction and a few applications

1. Self-duality

Ω: state space

$\eta(t), \xi(t), t \geq 0$: Two copies of a Markov process on Ω

$D : \Omega \times \Omega \to \mathbb{R}$: Duality function

Def The process is self-dual $\Leftrightarrow$

$$\mathbb{E}_\eta D(\eta(t), \xi) = \mathbb{E}_\xi D(\eta, \xi(t))$$

where $\eta = \eta(0), \xi = \xi(0)$.

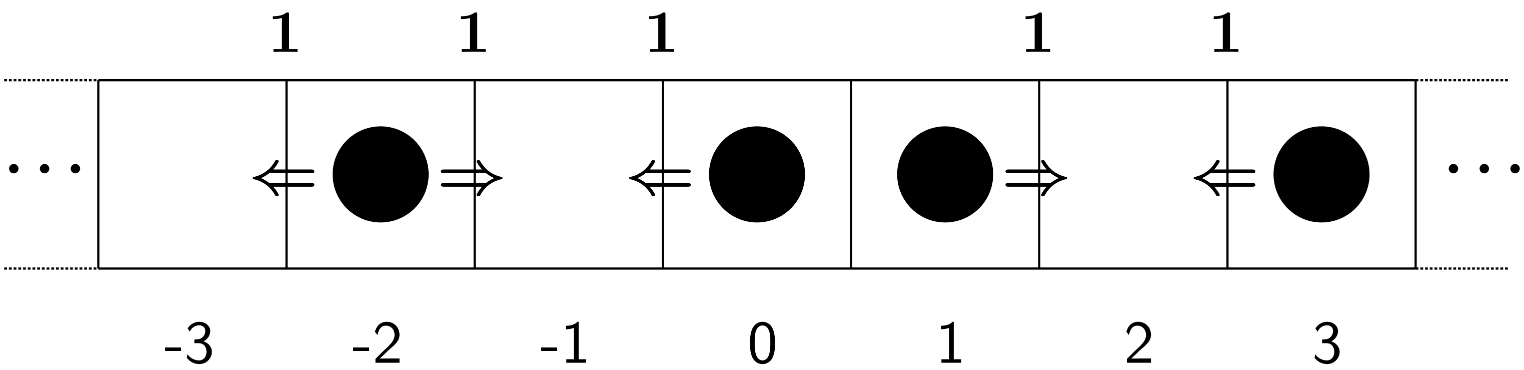
L: the generator of the Markov process

(For finite state space self-duality is equivalent to $LD = D^t L$.)

SEP

Symmetric simple exclusion process (SEP or SSEP)

1D case



$\eta_j = 1$ if site j is occupied, $\eta_j = 0$ if site j is empty.

Generator

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} (\eta_j(1-\eta_{j+1}) + (1-\eta_j)\eta_{j+1})[f(\eta^{j,j+1}) - f(\eta)]$$

Self-duality for SEP

- In Liggett it is stated as

$$\mathbb{P}_\eta[\eta(t) = 1 \text{ on } A] = \mathbb{P}_A[\eta = 1 \text{ on } A_t]$$

where $A = \{x_1, \ldots, x_m\}, x_1 < \ldots < x_m, m \in \mathbb{N}$.

- LHS is the m point correlation function $\mathbb{E}[\prod_{i=1}^m \eta_{x_i}(t)]$ for initial config η. RHS is the prob. that m particles starting from A are at m sites of η.

- This implies that m-point correlation functions of SEP satisfy the m-particle SEP dynamics. For example for $m = 1$

$$\frac{d}{dt}\mathbb{E}\eta_x(t) = \mathbb{E}\eta_{x-1}(t) + \mathbb{E}\eta_{x+1}(t) - 2\mathbb{E}\eta_x(t)$$

Matrix representation for finite SEP

- For finite SEP with L sites, $\Omega = \{0,1\}^L$ (finite state space).
- Duality function

$$D(\eta,\xi) = \prod_{i=1,\xi_i=1}^{L} \eta_i$$

- The adjoint generator ${}^tL_{\text{SEP}}$ of SEP

$${}^tL_{\text{SEP}} = \frac{1}{2}\sum_{j=1}^{L-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z - 1)$$

where $\sigma^{x,y,z}$ are Pauli matrices ($i = \sqrt{-1}$)

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- For these one can check $LD = D\,{}^tL$.

SU(2) symmetry

- In quantum statistical mechanics, the matrix ${}^t\boldsymbol{L}_{\mathsf{SEP}}$ is also known as the Hamiltonian of the Heisenberg chain $(= \boldsymbol{H}_{\mathsf{Hei}})$.
- $\boldsymbol{SU(2)}$ algebra

$$[\boldsymbol{S}^z, \boldsymbol{S}^\pm] = \pm \boldsymbol{S}^\pm$$
$$[\boldsymbol{S}^+, \boldsymbol{S}^-] = 2\boldsymbol{S}^z$$

- Casimir element: $\boldsymbol{J} = \boldsymbol{S}^+\boldsymbol{S}^- + \boldsymbol{S}^-\boldsymbol{S}^+ + 2\boldsymbol{S}^z\boldsymbol{S}^z$ commutes with $\boldsymbol{S}^\pm, \boldsymbol{S}^z$.
- The spin-$\frac{1}{2}$ representation is written using Pauli matrices: $\boldsymbol{S}^\pm = \frac{1}{2}(\boldsymbol{\sigma}^x \pm i\boldsymbol{\sigma}^y), \boldsymbol{S}^z = \frac{1}{2}\boldsymbol{\sigma}^z$

- One can consider the tensor product representation for L spin-$\frac{1}{2}$ spins. Set

$$S^{\pm} = \frac{1}{2}\sum_{j=1}^{L}(\sigma_j^x \pm i\sigma_j^y), \quad S^z = \frac{1}{2}\sum_{j=1}^{L}\sigma_j^z$$

They satisfy the $SU(2)$ algebra.

- The Casimir becomes H_{Hei} in the tensor product rep. and hence it commutes with these generators:

$$[H_{\text{Hei}}, S^{\pm}] = [H_{\text{Hei}}, S^z] = 0$$

- The self-duality of SEP is a consequence of this symmetry (with $D = e^{S^+}$).

Derivation of the self-duality relation

With $\langle N| = \langle 0|(S^+)^N/N!$ and $|I_N\rangle$: the initial state

$$
\begin{aligned}
&\langle \eta_{x_1}\cdots\eta_{x_m}\rangle \\
&= \langle N|\eta_{x_1}\cdots\eta_{x_m} e^{Ht}|I_N\rangle \\
&= \langle x_1,\cdots x_m|\frac{(S^+)^{N-m}}{(N-m)!}e^{Ht}|I_N\rangle
\end{aligned}
$$

[Comute S^+ with H]

$$
= \sum_{1\le z_1<\cdots<z_m\le L} \langle x_1,\cdots x_m|e^{Ht}|z_1,\cdots z_m\rangle\langle N|\eta_{z_1}\cdots\eta_{z_m}|I_N\rangle
$$

In the last equality, we use

$\mathbf{1} = \sum_{1\le z_1<\cdots<z_m\le L}|z_1,\cdots z_m\rangle\langle z_1,\cdots z_m|$

2. SU(1,1)

$SU(1,1)$ algebra

$$[K^0, K^\pm] = \pm K^\pm$$

$$[K^-, K^+] = 2K^0$$

Casimir: $C = K^+K^- + K^-K^+ - 2K^zK^z$

A representation

$$K^+ = \frac{1}{2}x^2$$

$$K^- = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$K^0 = \frac{1}{4}\left(\frac{\partial}{\partial x}x + x\frac{\partial}{\partial x}\right)$$

Brownian energy process

We consider the tensor product representation of $SU(1,1)$. The corresponding generator is given by

$$L = -4 \sum_j L_{j,j+1}$$

with

$$L_{j,j+1} = K_j^+ K_{j+1}^+ + K_j^- K_{j+1}^- - 2K_j^0 K_{j+1}^0 + 1/2$$
$$= \left(x_j \frac{\partial}{\partial x_{j+1}} - x_{j+1} \frac{\partial}{\partial x_j} \right)^2$$

$L_{j,j+1}$ conserves the energy $x_j^2 + x_{j+1}^2$ and generates a Brownian rotation of the angle $\arctan(x_{j+1}/x_j)$.

The dynamics of x_i^2 is called the Brownian energy process(BEP).

k-BEP

Another representation of $SU(1,1)$ with parameter k

$$K^+ = \frac{1}{2} z$$
$$K^- = 2z\partial_z^2 + 4k\partial_z$$
$$K^0 = z\partial_z + k$$

For this (with $\partial_j = \partial_{z_j}$)

$$L_{j,j+1} = K_j^+ K_{j+1}^+ + K_j^- K_{j+1}^- - 2K_j^0 K_{j+1}^0 + k^2/2$$
$$= z_i z_j \left(\partial_j - \partial_{j+1}\right)^2 - 2k(z_j - z_{j+1}) \left(\partial_j - \partial_{j+1}\right)$$

$k = 1/2$ case corresponds to the usual BEP (with $z = x^2$).

BEP can also be obtained as a limiting case of a particle system.

Symmetric Inclusion Process(SIP)

By considering the tensor product of another discrete representation of $SU(1,1)$ with parameter $k(\geq 0)$, one can construct a process, SIP(k), with generator (with $\eta_i \in \mathbb{N}$)

$$(L^{SIP(k)}f)(\eta) := \sum_{i=1}^{L-1} (L^{SIP(k)}_{i,i+1} f)(\eta) \qquad \text{with}$$

$$(L^{SIP(k)}_{i,i+1} f)(\eta) :=$$

$$= (\eta_i(k+\eta_{i+1}) + (k+\eta_i)\eta_{i+1})(f(\eta^{i,i+1}) - f(\eta))$$

Prop. This process has a self-duality related to $SU(1,1)$.

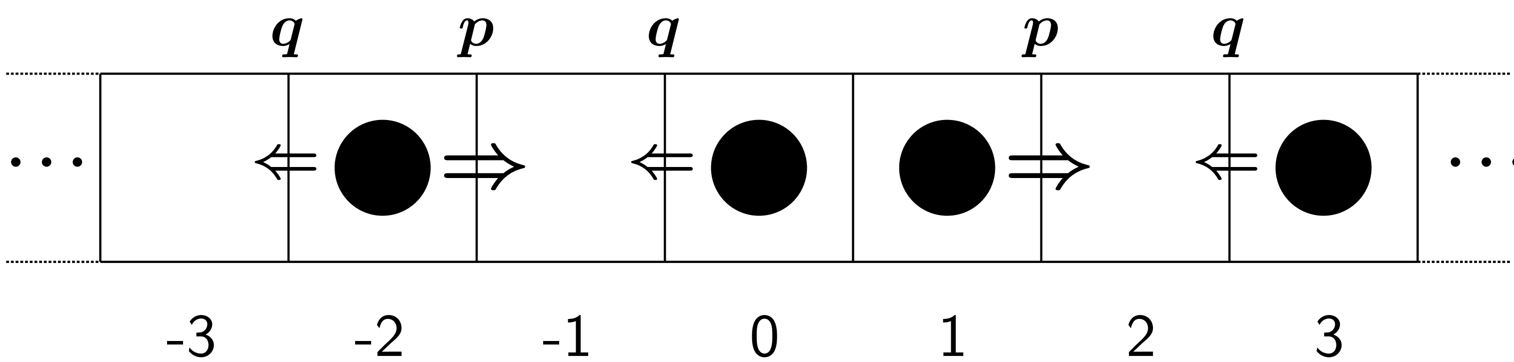
Prop. In a diffusion scaling limit, this tends to k-BEP.

KMP model

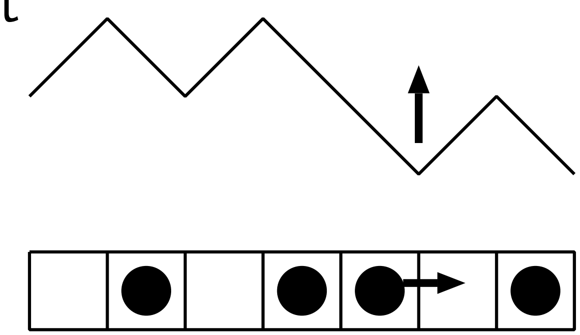
- KMP(Kipnis-Marchioro-Pressutti) model
- A bond $(i, i+1)$ is randomly selected and the energies of the two sites $i, i+1$ are uniformly redistributed under the constraint of conservation of $E_i + E_j$.
- KMP is the "instantaneous thermalization" limit of BEP.
- This is one of the few models for which one can do concrete analysis about fluctuations.

3. ASEP

ASEP $=$ asymmetric simple exclusion process



- SEP($p = q$), TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x, t)$: Integrated current at $(x, x + 1)$ upto time t
- In a certain weakly asymmetric limit
  ASEP $\Rightarrow$ KPZ equation



Self-duality

- 1997 Schütz

  The n-point function of the form $\mathbb{E}[\prod_{i=1}^{n} q^{N(x_i,t)}]$ satisfies the n particle dynamics of the same process (self-duality).

- The adjoint generator of ASEP is equivalent to the Hamiltonian of XXZ spin chain by a similarity transformation.

  The self-dality is related to $U_q(sl_2)$ symmetry of XXZ and ASEP.

- 2012-2015 Borodin-Corwin-TS

  The self-duality of ASEP can be used to study the fluctuations of current $N(x,t)$.

Deformed algebra $U_q(sl_2)$

$$[J^+, J^-] = [2J^0]_q, \qquad [J^0, J^\pm] = \pm J^\pm$$

and

$$[2J^0]_q := \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}$$

Casimir element

$$C = J^- J^+ + [J^0]_q [J^0 + 1]_q$$

Tensor product representation (with a deformed co-product).

XXZ spin chain

By considering the tensor product representation of L spin-$\frac{1}{2}$ spins, we see that the XXZ spin chain Hamiltonian with boundary magnetic fields

$$H_{\mathrm{XXZ}} = h\sigma_1^z + \frac{1}{2}\sum_{j=1}^{L-1}[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)] - h\sigma_L^z$$

with $h = (Q - Q^{-1})/4, \Delta = (Q + Q^{-1})/2$ has the $U_Q(sl_2)$ symmetry.

ASEP and XXZ

Adjoint generator of ASEP (with reflective bounaries)

$$ {}^{t}L_{\mathrm{ASEP}} = \sum_j \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{j,j+1} $$

With $Q = \sqrt{q/p}, \Delta = (Q + Q^{-1})/2$ and $V = \prod_j Q^{jn_j}$ where $n_j = \frac{1}{2}(1 - \sigma_j^z)$ this is related to XXZ hamiltonian by

$$ V\, {}^{t}L_{\mathrm{ASEP}} V^{-1} / \sqrt{pq} = H_{\mathrm{XXZ}} $$

4. A general construction

- H: $n \times n$ real symmetric matrix with non-negative off diagonal elements. The lowest eigenvalue is taken to be 0.
- By Perron-Frobenius theorem, there exist $g \in \mathbb{R}^n$ with strictly positive entries such that $Hg = 0$.
- Let us denote by G the diagonal matrix with entries $G(x,x) = g(x)$ for $x \in \Omega$.
- The matrix
$$L = G^{-1}HG$$
is a generator of a Markov process.
- If $[H,S] = 0$, then $[L, G^{-1}SG] = 0$ and $D = G^{-1}SG^{-1}$ is a self-duality function for the process with generator L.

Main results

By applying the general scheme in the previous slide to a deformed algebra, one can systematically try to construct Markov processes with asymmetry which has self-duality.

- By applying the scheme to $U_q(sl_2)$, one can construct a generalization of ASEP in which there could be more than one particles on each site.
- By applying the scheme to $U_q(su(1,1))$, one can construct a generalization of BEP and as a limiting case an asymmetric version of the KMP model.
- The scheme was applied to $U_q(sl_3)$ and $U_q(sp_4)$ by Kuan ($U_q(sl_3)$ also by Belitsky-Schütz).

Application 1: Spin j representation of $U_q(sl_2)$

The Markov process ASEP(q, j) on $[1, L] \cap \mathbb{Z}$ with closed boundary conditions is defined by the generator

$$(Lf)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1} f)(\eta) \qquad \text{with}$$

$$(L_{i,i+1} f)(\eta) = q^{\eta_i - \eta_{i+1} - (2j+1)} [\eta_i]_q [2j - \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) + q^{\eta_i - \eta_{i+1} + (2j+1)} [2j - \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))$$

Thm. This process has a self-duality related to $U_q(sl_2)$.

Rem 1: $j = 1/2$ is ASEP. $j = \infty$ corresponds to q-TASEP.

Rem 2: The process appeared in the talk by Shen. Convergence to the KPZ equation strongly suggests ASEP(q, j) is in KPZ class.

Application 2: $U_q(su(1,1))$

For $q \in (0,1)$ we consider the algebra with generators K^+, K^-, K^0 satisfying the commutation relations

$$[K^0, K^\pm] = \pm K^\pm, \qquad [K^-, K^+] = [2K^0]_q$$

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Casimir element

$$C = [K^0]_q [K^0 - 1]_q - K^+ K^-$$

Asymmetric process with self-duality

By considering the tensor product of a representation with parameter k, we can construct a process, ASIP(q, k), with closed boundary conditions with generator

$$(L^{ASIP(q,k)}f)(\eta) := \sum_{i=1}^{L-1} (L_{i,i+1}^{ASIP(q,k)}f)(\eta) \qquad \text{with}$$

$$\begin{aligned}(L_{i,i+1}^{ASIP(q,k)}f)(\eta) \\ := q^{\eta_i-\eta_{i+1}+(2k-1)}[\eta_i]_q[2k+\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ + q^{\eta_i-\eta_{i+1}-(2k-1)}[2k+\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))\end{aligned}$$

Thm. This process has a self-duality related to $U_q(su(1,1))$.

Asymetric Brownian Energy Process ABEP

Consider the limit of weak asymmetry $q = 1 - \epsilon\sigma \to 1$ $(\epsilon \to 0)$ combined with the number of particles proportional to ϵ^{-1}, going to infinity, and work with rescaled particle numbers $x_i = \lfloor \epsilon \eta_i \rfloor$.

Generator

Let $\boldsymbol{\sigma > 0}$ and $\boldsymbol{k \geq 0}$. The generator of ABEP$(\boldsymbol{\sigma, k})$ is

$$\boldsymbol{L^{ABEP^{(\sigma,k)}} f(x) = \sum_{i=1}^{L-1} [L_{i,i+1}^{ABEP^{(\sigma,k)}} f](x)}$$

with

$$\boldsymbol{L_{i,i+1}^{ABEP^{(\sigma,k)}} f(x) = \frac{1}{4\sigma^2} (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2}$$

$$\boldsymbol{- \frac{1}{2\sigma} \left\{(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})\right\}}$$

$$\boldsymbol{\times (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}) f(x)}$$

$\boldsymbol{\sigma \to 0}$ correspondes to $\boldsymbol{k}$-BEP.

Asymmetric version of the KMP model

By considering an "instantaneous thermalization" limit of the ABEP, we can define am asymmetric KMP with asymmetry parameter $\sigma \in \mathbb{R}_+$ as the process with generator given by:

$$L^{AKMP(\sigma)}f(x) = \sum_{i=1}^{L-1} \left\{ \frac{2\sigma(x_i + x_{i+1})}{e^{2\sigma(x_i+x_{i+1})} - 1} \right.$$

$$\cdot \int_0^1 [f(x_1, \ldots, w(x_i + x_{i+1}), (1-w)(x_i + x_{i+1}), \ldots, x_L) - f(x)$$

$$\left. \times\ e^{2\sigma w(x_i+x_{i+1})}\, dw \right\}$$

- This is an example with duality but without integrability.
- Properties of the process are yet to be studied.

Non-zero current for ABEP

For $\mathbb{E}$ the expectation wrt translation invariant stationary measure,

$$\frac{d}{dt}\mathbb{E}[x_i(t)] = J_{i-1,i} - J_{i,i+1}$$

where $J_{i,i+1} := -\mathbb{E}[\Theta_{i,i+1}]$ with

$$\Theta_{i,i+1}(x)$$
$$= -\frac{1}{2\sigma}\left\{(1 - \mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}} - 1) + 2k(2 - \mathrm{e}^{-2\sigma x_i} - \mathrm{e}^{2\sigma x_{i+1}})\right\}$$

Prop.

$$J_{i,i+1} = -\mathbb{E}[\Theta_{i,i+1}] < 0 \qquad \text{if} \quad k > 1/2$$

$$J_{i,i+1} = -\mathbb{E}[\Theta_{i,i+1}] > 0 \qquad \text{if} \quad k = 0$$

Proof

In the case $k > 1/2$, we obtain

$$\mathbb{E}[\Theta_{i,i+1}] = \frac{1}{2\sigma}\left\{(1-4k)+(2k-1)\mathbb{E}(\mathrm{e}^{2\sigma x_{i+1}}+\mathrm{e}^{-2\sigma x_i})+\mathbb{E}(\mathrm{e}^{2\sigma(x_{i+1}-x_i)})\right\}$$

Since expectation in the translation invariant stationary state of local variables are the same on each site and $\cosh(x) \geq 1$ one obtains

$$\mathbb{E}[\Theta_{i,+1}] \geq \frac{1}{2\sigma}\left\{(1-4k)+2(2k-1)+\mathbb{E}[\mathrm{e}^{2\sigma(x_{i+1}-x_i)}]\right\}$$

Furthermore, Jensen inequality and translation invariance implies that

$$\mathbb{E}[\Theta_{i,i+1}] > \frac{1}{2\sigma}\left\{(1-4k)+2(2k-1)+1\right\} = 0$$

In the case $\boldsymbol{k = 0}$ one has

$$\mathbb{E}[\boldsymbol{\Theta_{i,i+1}}] = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\sigma}}\,\mathbb{E}\Big[(\mathbf{1} - \mathbf{e}^{-\mathbf{2}\boldsymbol{\sigma x_i}})(\mathbf{1} - \mathbf{e}^{\mathbf{2}\boldsymbol{\sigma x_{i+1}}})\Big] < \mathbf{0}$$

which is negative because the function is negative a.s.

Summary

- (Self-)duality: The m-point correlation function can be reduced to m-particle problem
- Self-dualities for asymmetric processes. Current fluctuations for ASEP
- A general scheme to construct Markov processes with (deformed) symmetry
- Examples of spin $U_q(sl_2)$ and $U_q(su(1,1))$.
- Properties of ASIP and the asymmetric KMP model?