From Riemann, Hodge, Chern, Kodaira and Hirzebruch to Modern Development on Complex Manifold

Shing-Tung Yau

Harvard University

Riemann conference, Sanya

December 27, 2015

The work of Riemann and Kodaira

Riemann was one of the founders of complex analysis, along with Cauchy. Riemann pioneered several directions in the subject of holomorphic functions:

1. The idea of using differential equations and variational principle. The major work here is the Cauchy-Riemann equation, and the creation of Dirichlet principle to solve the boundary value problem for harmonic functions. (It took several great mathematicians, such as David Hilbert, to complete this work of Riemann.)

2. He gave the proof of the Riemann mapping theorem for simply connected domains. This theory of uniformization theorems has been extremely influential. There are methods based on various approaches, including methods of partial differential equations, hypergeometric functions and algebraic geometry. A natural generalization is to understand the moduli space of Riemann surfaces where Riemann made an important contribution by showing that it is a complex variety with dimension $3g - 3$.

3. The idea of using geometry to understand multivalued holomorphic functions, where he looked at the largest domain that a multivalued holomorphic function can define. He created the concept of Riemann surfaces, where he studied their topology and their moduli space. In fact, he introduced the concept of connectivity of space by cutting Riemann surface into pieces. The concept of Betti number was introduced by him for spaces in arbitrary dimension.

The idea of understanding analytic problems through topology or geometry has far-reaching consequences. It influenced the later works of Poincaré, Picard, Lefstchetz, Hodge and others. Important examples of Riemann's research is to use monodromy groups to study analytic functions. Such study has deep influence on the development of discrete groups in the 20th century. The Riemann-Hilbert problem was inspired by this and up to now, is still an important subject in geometry and analysis. The study of ramified covering and the Riemann-Hurwitz formula gave an efficient technique in algebraic geometry and number theory.

4. The discovery of Riemann-Roch formula over algebraic curve. The formula gave an effective way to calculate the dimension of holomorphic sections of a holomorphic bundle. The generalizations by Kodaira, Hirzebruch, Grothendieck, Atiyah-Singer have led to tremendous progress in mathematics in the twentieth century.

5. His study of period integrals related to Abel-Jacobi map and the hypergeometric equations:

$$z(1-z)y'' + [c - (a+b+c)z]y' - aby = 0.$$

6. The study of Riemann bilinear relations, the Riemann forms and the theta functions. During his study of the periods of Riemann surfaces, he found that the period matrix must satisfy period relations with a suitable invertible skew symmetric integral matrix which is called Riemann matrix later. Riemann realized that the period relations give necessary and sufficient condition for the existence of non-degenerate Abelian functions.

(According to Siegel, his formulation was incomplete and he did not supply a proof. Later, Weierstrass also failed to establish a complete proof despite his many efforts in this direction. Complete proofs were finally attained by Appell for the case $g = 2$ and by Poincaré for arbitrary g.)

It should be noted that Riemann spent the last four years of his life in Italy because he contracted Tuberculosis and needed to avoid the severe winter in Germany. But as a result, he inspired a large group of differential geometers and projective algebraic geometers in Italy. Their works influenced the development of geometry and physics in the 20th century.

First of all, we should say that Riemann was the mathematician that brought us a new concept of space that was not perceived by any mathematician before him. I believe that was the reason that Gauss was so touched by his famous address on the foundations of geometry in 1854. I could not read German and was only able to read this address recently after it was translated into English. I was rather surprised that Riemann had rather liberal view about what geometry is supposed to be.

His guiding principle was nature itself:

> *The theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally .... This takes us into the realm of another science – physics.*

B. Riemann ("On the Hypotheses Which Lie at the Foundation of Geometry," 1854)

He thinks a deep understanding of geometry should be based on concepts of physics. And this is indeed the case as we experienced in the past century, especially in the past 50 years development of geometry. Although he was the one who introduced the concept of Riemann surface, which is the largest domain that a multivalued holomorphic function lives in, the precise modern concept was developed much later through the efforts of Klein, Poincaré and others.

While Felix Klein already used atlas to describe Riemann surface, it has to wait until Hermann Weyl who first gave the modern rigorous definition of Riemann surface, in terms of coordinate charts.

It was rather strange that a formal introduction of the concept of complex manifold was quite a bit later. Historically, generalization of one complex variable to several complex variables began by the study of functions on domains in $\mathbb{C}^n$. There were fundamental works of Levi, Oka, and Bergman.

The natural generalization of the concept of two dimensional surfaces to higher dimensional manifolds was done by O. Veblen and J.H.C. Whitehead in 1931-32. H. Whitney (1936) clarified the concept by proving that differentiable manifolds can be embedded into Euclidean space.

However, it was only in 1932 at the International Congress of Mathematicians in Zurich, did Caratheodory study "four dimensional Riemann surface" for its own sake. In 1944, Teichmüller mentioned "komplexe analytische Mannigfaltigkeit" in his work on "*Veränderliche Riemannsce Flächen*".

Chern was perhaps the first one to call the English "complex manifold" in his work.

The general abstract concept of almost complex structure was introduced by Ehresmann and Hopf in the 1940s. In 1948, Hopf proved that the spheres S^4 and S^8 cannot admit almost complex structures.

The concept of Kähler geometry was introduced by Kähler in 1933 where he demanded the Kähler form (which was first constructed by E. Cartan) to have a Kähler potential. Kähler had already observed special properties of such metric. He knew that the Ricci tensor associated to the metric tensor $g_{i\bar{j}}$ can be written rather simply as

$$R_{k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l}(\log \det g_{i\bar{j}}),$$

which gave a globally defined closed form on the manifold.

He knew that it defines a topological invariant for the geometry. It defines a cohomology class independent of the metric. It was found later that, after normalization, it represents the first chern class of the manifold. The simplicity of the Ricci form allows Kähler to define the concept of Kähler-Einstein metric and he wrote down the equation locally in terms of the Kähler potential. He gave examples of the Kähler metric of the ball.

Slightly afterwards, Hodge developed Hodge theory, without knowing the work of Kähler, based on the induced metric from projective space to the algebraic manifolds. He studied the theory of harmonic forms with special attention to algebraic manifolds. The (p, q) decomposition of the differential forms have tremendous influence on the global understanding of Kähler manifolds. A very important observation is that the Hodge Laplacian commutes with the projection operator to the (p, q)-forms and hence the (p, q) decomposition descends to the de Rham cohomology. The theory was soon generalized to cohomology with twisted coefficients.

A very important cohomology with twisted coefficient is cohomology with coefficient in the tangent bundle or cotangent bundle, and their exterior powers. For the first cohomology with coefficient in tangent bundle, Kodaira and Spencer developed the fundamental theory of deformation of geometric structures, which gave far reaching generalization of the works of Riemann, Klein, Teichmüller and others on parametrization of complex structures over Riemann surfaces.

They realize that the first cohomology with coefficient on tangent bundle, denoted by $H^1(T)$, parametrize the complex structure infinitesimally and that the second cohomology with coefficient on tangent bundle, denoted by $H^2(T)$, gives rise to obstruction to the deformation. The last statement was made very precisely by Kuranishi using Harmonic theory of Hodge-Kodaira. It describes the singular structure of the moduli space locally.

Kodaira-Spencer studied how elements in $H^1(T)$ acts on other cohomology, which leads to study of variation of Hodge structures. The Hodge groups can be grouped in an appropriate way to form a natural filtration of the natural de Rham groups. The Kodaira-Spencer map plays a very important role in understanding the deformation of such filtrations. Cohomology with coefficients of cotangent bundle or wedge product of cotangent bundle gives to Hodge (p, q) forms. The duality of tangent bundle and cotangent bundle gives rise to something called mirror symmetry studied extensively in the last thirty years in relation to the theory of Calabi-Yau manifolds.

A very important tool in complex geometry was the introduction of Chern classes to complex bundles over a manifold and the representation of such classes by curvature of the bundle.

When Chern introduced the concept of Chern class, he was influenced by the work of Pontryagin on characteristic classes. In the course of defining Chern classes by de Rham forms given by symmetric polynomial of the curvature form, Chern developed the Chern connections for holomorphic bundles. It was also Chern who proved that Chern classes for algebraic manifolds are represented by algebraic cycles. This has been the major evidence of the famous Hodge conjecture, which says that every (p, p)-class of an algebraic manifold can be represented by algebraic cycles.

Chern proved that there are three different approaches to define Chern classes and explained why they are integral classes. Weil explained how they are related to Ad-invariant polynomials. It was remarkable that Weil made a remark that the integrality of Chern class should mean that it will play a role in quantum theory. Chern-Weil theory forms a bridge between topology and geometry.

The desire to generalize Riemann-Roch formula to higher dimensional algebraic manifolds has been relative slow, until the very powerful methods of sheaf theory was introduced by Leray, and important inputs were given by Weil, Borel and Serre. These basic techniques enabled Hirzebruch to arrive at the important Hirzebruch-Riemann-Roch formula in his 1954 paper, which can be stated in the following way:

$$\chi(V, E) = \int_V \mathrm{ch}(E)\mathrm{td}(V),$$

where E is a holomorphic vector bundle over a projective variety V.

The formulation of this formula by itself is remarkable: Hirzebruch developed the splitting principle and the theory of multiplicative sequences to find a formal power series of Chern classes. The Todd class is such a power series which is found by Hirzebruch to represent the arithmetic genus of an algebraic manifold. The Chern character was invented by him to be a homomorphism from space of algebraic bundles to even dimensional cohomology .

In the other direction, Kodaira was the first major mathematician who developed Hodge theory of harmonic forms right after its announcement by Hodge, and he generalized the theory of harmonic forms to manifolds with boundaries, where various boundary conditions have to be imposed.

Perhaps his most important work was his deep understanding that the Bochner argument in Riemannian geometry can be used to prove a vanishing theorem for cohomology classes under curvature condition of the manifold. He realized that the natural place for such vanishing theorem is to deal with cohomology with coefficient on bundle or sheaf. The vanishing theorem of Kodaira says that for positive line bundle L on a compact complex manifold M:

$$H^q(M, K_M \otimes L) = 0$$

for $q > 0$.

Coupled with the following theorem of Serre duality:

$$H^q(M, E) \cong H^{n-q}(M, K \otimes E^*),$$

Kodaira vanishing theorem implies that the Euler characteristic of cohomology with coefficients in a holomorphic vector bundle E with $E \otimes K^*$ positive, is simply the dimension of the group of holomorphic sections of E.

The above mentioned Hirzebruch-Riemann-Roch theorem then gives a formula to compute the dimension of the sections of the holomorphic bundle in terms of Chern numbers defined by Chern classes of the manifold and the bundle. This creates the most basic tool to understand algebraic manifolds.

Kodaira also showed that by blowing up points on the manifold, one can find enough holomorphic sections to separate points of the original manifolds and in fact gives an embedding of the manifold into complex projective space by using holomorphic sections of a very ample line bundle.

$$M \to \mathbb{P}^N, \qquad p \mapsto [s_0(p), \ldots, s_N(p)]$$

where $s_0, \ldots, s_N$ is a basis for $H^0(M, L)$.

In particular, he proved that any Kähler manifold, whose Kähler class is defined by the Chern class of a holomorphic line bundle, can be holomorphically embedded into the complex projective space. The theorem of Chow then implies the manifold is in fact defined by an ideal of homogeneous polynomials, and hence an algebraic manifold.

What Kodaira has proved is one of the most spectacular theorems in mathematics, and a glorious generalization of the work of Riemann on the condition of a complex torus to be abelian. More importantly the method of proving the Kodaira vanishing theorem has far reaching consequences in complex geometry. It was generalized to noncompact complex manifold, by various mathematicians including C. Morrey, Hörmander, Kohn, Vesentini, and others.

However, an upper bound of the power of the line bundle is not clear from Kodaira's argument.

Later on, Matsusaka (improved by Kollár-Matsusaka) proved the very-ampleness of mL for an ample line bundle L on an n-dimensional projective variety X, when m is no less than a bound, depending only on the intersection numbers L^n and $K_X \cdot L^{n-1}$ on X.

In 1980s, Kawamata proved his famous basepoint freeness theorem about the pluricanonical systems of minimal models. This is very important in the study of abundance conjecture. He proved that under the assumption that the numerical Kodaira dimension of a minimal variety X is equal to its Kodaira dimension, the pluricanonical system $|mK_X|$ is basepoint free for large m. This implies the basepoint freeness for minimal models of general type varieties. Later on, in a series paper of Miyaoka and Kawamata, they settled the proof of abundance conjecture for threefolds.

An important unsolved conjecture was proposed by Fujita in 1985, $mL + K_X$ is base-point free for $m \geq n + 1$ and is very ample for $m \geq n + 2$. Many mathematicians did important work on Fujita's conjecture, including Reider, Ein-Lazarsfeld, Kawamata, and many others. Demailly proved an effective formula for the bound on very ampleness. Angehrn and Siu proved a quadratic bound for basepoint freeness.

There are many other contributioins to algebraic geometry made by Japanese algebraic geometers. Mori first introduced the ingenious idea of “bend and break” argument in his proof of Hartshorne conjecture. This leads to his proof of cone theorem in birational geometry and had deep influences in minimal model program. Mukai introduced the Fourier-Mukai transform in 1981. This became an important tool in the study of derived categories.

Calabi conjecture and Kähler-Einstein metrics

The theorems by Kodaira, Matsusaka and Kawamata provide abundance of holomorphic sections for the holomorphic line bundle to embed the manifold into complex projective with higher dimension.

An interesting important problem is the zero codimension case where we want to embed X to complex projective space with the same dimension. This can be interpreted as a generalization of uniformization theorem from 2-dimensional sphere to higher dimension.

Hirzebruch and Kodaira proved that every algebraic manifold that is homeomorphic to $\mathbb{CP}^n$ is actually biholomorphic to it. They used Hirzebruch-Riemann-Roch formula, but they could only treat the case of odd dimensional manifolds due to the indeterminacy of the sign of the first Chern class. The even dimensional case was finally settled by me in 1976.

My argument depends on the existence of Kähler-Einstein metrics assuming the first Chern class is either positive, zero or negative. Although the Kähler-Einstein metric was already discussed by Kähler in his 1933 paper, where he wrote the equation explicitly, it wasn't until 1954 when Calabi made a formal proposal to prove the existence of Kähler metric with prescribed volume form.

This could be used to proved the existence of Ricci-flat Kähler metric for any polarization if the first Chern class of the manifold is zero. Then Calabi asked the question when the first Chern class of the manifold is either negative or positive. The questions of Calabi were believed to be too good to be true in the old days, as nobody was able to construct an explicit Kähler-Einstein metric on any compact Kähler manifolds with no symmetries.

In 1976, I settled the cases when the first Chern class is either trivial or negative. (Aubin did the work independently for negative first Chern class.) I also considered the case when the manifold can have singularities, as was announced in my talk at 1978 ICM in Helsinki.

Kähler-Einstein metrics on Fano manifolds

When the first Chern class is positive, it is called a Fano manifold. There are many interesting properties about Fano manifolds. Kollár, Mori and Miyaoka showed that smooth Fano varieties are rationally connected, in the sense that any two points are connected by a rational curve with (effectively) bounded degree. This implies an effective bound for the degree of the Fano n-fold, with respect to its anti-canonical bundle. Based on the work of Kollár and Matsusaka, it also implies that Fano n-folds form a bounded family.

In this case, there is an obstruction for the existence of Kähler-Einstein metric due to Matsushima: the Lie algebra of the automorphism group of the manifold must be reductive. Futaki introduced his beautiful invariant defined on this Lie algebra. The Futaki invariant soon became a fundamental tool to study Kähler-Einstein metric on Fano manifolds.

On the other hand, It took a long while to find a necessary and sufficient condition for the existence of Kähler-Einstein metric on Fano manifolds. Many people, including Calabi, was misled to believe that the non-existence of nonzero holomorphic vector fields is enough for the existence of Kähler-Einstein metric on Fano manifolds.

Right after I proved the Calabi conjecture on the existence of Kähler metric with prescribed volume form, I tried to work on the problem of the existence of Kähler-Einstein metric on Fano manifolds.

It is clear that based on the (nontrivial) higher order estimates that I had (independently due to Aubin for second order estimate) in the proof of the Calabi conjecture, the only missing point is some integral estimate of the Kähler potential. I found it is useful to set up the continuity argument

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = \exp(h - tu) \det(g_{i\bar{j}}),$$

where $t = 0$ correspond to a Kähler metric with positive Ricci curvature, as was given by the Calabi conjecture.

A simple calculation shows that the Ricci curvature of all members in the family have positive lower bound. This simplifies the analysis quite a bit as we have experiences with compact manifolds with Ricci curvature bounded from below by positive constant. In 1978, I returned to Stanford from my visit of Berkeley. At that time, I succeeded to convince Stanford mathematics department to hire Y.-T. Siu to come to Stanford from Yale.

We started to think about a proof of the existence of Kähler-Einstein metric by finding some integral estimate of the Kähler potential. Many estimates were found, but they are short of proving the existence of the metric. Some of those estimates can be sharpened if there are symmetries on the manifold, a procedure similar to the way that Moser sharpened the Trudinger inequality on the sphere when there is antipodal symmetry.

In the meanwhile, I realized that Bogomolov used the concept of stability of bundles to prove Chern number inequalities for algebraic surfaces which were sharpened by Miyaoka and myself independently. I started to believe there has to be links between the concept of stability with the existence of Hermitian Yang-Mills connections on bundles.

The fact that a holomorphic bundle admits a Hermitian Yang-Mills connection if and only if the bundle is polystable was proved by Uhlenbeck-Yau in arbitrary compact Kähler manifolds, and by Donaldson in 2-dimension. Simpson observed that the proof of Uhlenbeck-Yau can be used to settle the case when there is a Higgs field. (Up to now, the Uhlenbeck-Yau argument is in fact the only argument to prove such a statement.)

The Bogomolov inequality is optimal for general stable bundles. But it is not as sharp as the Miyaoka-Yau inequality when applied to the tangent bundle of the manifold. Hence I suspected that existence of Kähler-Einstein metric should be considered as a nonlinear version of the existence of Hermitian Yang-Mills connection, and the stability of bundle should be replaced by manifold stability. Therefore only in early 1980s, I realized that the right condition for existence of Kähler-Einstein metric is the stability of the algebraic manifold.

I made the conjecture that the existence of Kähler-Einstein metric is equivalent to stability. I told all my graduate students about this conjecture, especially to Gang Tian who showed interest in the problem of Kähler-Einstein metric. But it took a long time to convince him of the validity of my conjecture.

There are many ways to define stability of manifolds including the concepts of Chow stability or Hilbert stability. I was not sure which one is correct. But I started to explore it with my students in my seminars. First of all, one had to make sure that algebraic stability, which is defined by embeddings of algebraic manifolds into complex projective space, can be linked to existence of Kähler-Einstein metric.

In fact, in order to link stability condition to algebraic geometry, I proposed to prove any Hodge metric on an algebraic manifold can be approximated by normalized Fubini-Study metric induced on the manifold through embedding of the manifold into complex projective space by high powers of an ample line bundle.

I asked Tian to follow this line of argument to finish the first step of my conjecture on the equivalence of stability of Fano manifolds with the existence of Kähler-Einstein metrics.

I suggested Tian to use my method with Siu on the uniformization of Kähler manifolds to produce peak functions to achieve such a goal. (The purpose of that paper with Siu was also embedding of Kähler manifolds.)

The proof was reasonably transparent using technology from my paper with Siu. This became Tian's thesis at Harvard.

The method can be said to be an understanding of the works of Kodaira in the analytic setting. The work was carried out as I expected and it was strengthened by Catlin, Zelditch and by Lu.

So, we know that we can approximate any Hodge metric by the induced metric of the projective embedding of the manifold into some complex projective spaces. However, there is an ambiguity due to the action of complex projective group. This is of course what geometric invariant theory studies.

It turns out that when I studied first eigenvalue of the Laplacian with Bourguignon and Peter Li, we need to find a good position for the embedding upon action by the projective group, which we called the balanced condition. It can be written in the following form:

$$\int_{\sigma(M)} \frac{z_i \bar{z}_j}{|z_0|^2 + \cdots + |z_N|^2} \omega_{FS}^n = \frac{\mathrm{vol}(M)}{N+1} \delta_{ij}$$

for some $\sigma \in SL(N+1, \mathbb{C})$

With such a condition, we can use the embedding to give a good estimate of the first eigenvalue in terms of the total volume and the degree defined by the Chern form wedge with the Kähler classes.

I suggested this condition as a starting point to my former student Luo to understand the concept of stability required to prove my conjecture on the existence of Kähler-Einstein metric based on stability. A theorem of Shouwu Zhang says that the existence of a unique balanced embedding is equivalent to the manifold being Chow-Mumford stabie.

Luo found it effective to change the measure in the above formula defined by the induced measure of the ambient projective space. And it turns out that for a polarized manifold (M, L) if there exists a metric on L such that the Bergman function of L^k is constant for some k, then it is Chow stable.

My conjecture that the existence of Kähler-Einstein metric is equivalent to stability was announced several times in several conferences and was explicitly written in my article for the proceedings of UCLA conference on differential geometry in 1990.

I also communicated to Tian in detail on how to understand the Futaki invariant in this setting. The final conjecture of mine was solved recently by Chen-Donaldson-Sun based on earlier works of Donaldson including the right algebro-geometric definition of K-stability.

According to Donaldson, a Fano manifold is called K-stable if all its non-trivial test configurations (which describe certain degeneration of Kähler manifolds by flat families) have positive Futaki invariants. For a test configuration $\mathcal{X} \to \mathbb{C}$ with $\mathbb{C}^*$ action, the Futaki invariant F_1 can be found from the total weight w_k of $\mathbb{C}^*$ acting on $H^0(X_0, L^k)$, using

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + O(k^{-2})$$

where d_k is the dimension of $H^0(X_0, L^k)$.

But the condition of K-stability is not easy to check, even in the case of surfaces. It would therefore be interesting to prove the existence of balanced condition for high power embeddings of a Fano manifold implies existence of Kähler-Einstein metrics. It is highly desirable to clarify the condition of K-stability so that it can be checked effectively.

Hermitian-Yang-Mills connections

Hermitian metric on a complex manifold has a natural generalization to Holomorphic bundles over complex manifolds. Given a Hermitian metric on the bundle, there is a natural connection which preserves the metric and also the $(0, 1)$ part of the covariant derivative would be the same as the naturally defined $\bar{\partial}$ operator that depends only on the complex structure of the bundle and the complex manifold. The curvature is a $(1, 1)$-form with values in the endomorphism of the bundle.

There is a natural generalization of the Kähler-Einstein condition to this setting by wedging the curvature 2-form with the Kähler form to the top dimension and require it to be a scalar multiple of identity tensor with the volume form. This equation is the natural generalization of anti-self-dual equations for bundles over a Kähler surface.

In fact, around 1974, C.N. Yang was trying to solve the anti-self-dual Yang-Mills equation on $\mathbb{R}^4$, and he showed that it can be reduced to Cauchy-Riemann equations. And therefore he demonstrated that the above equation is part of Yang-Mills equations. It is therefore natural to call such connection to be Hermitian Yang-Mills connection.

The equation became rather well known in the math community after 1976, when people recognized the importance of applications of Kähler-Einstein metric to complex geometry. The proof of the existence of such connections would be clearly different as the Calabi-Yau theorem was based on the complex Monge-Amperè equation which depends only on a scalar. The Hermitian Yang-Mills connection is a vector-valued equation. In 1977, I discussed with Siu on this problem, but with no result, as it was not clear what the obstruction would be.

In 1978, when I gave the talk at 1978 ICM in Helsinki, I thought more about the possible obstructions. I conclude that it has to be related to the slope stability of the holomorphic bundle as was motivated by the work of Bogomolov and Miyaoka on Chern number inequalities. I was informed that this was also observed by Hitchin and Kobayashi independently.

Donaldson-Uhlenbeck-Yau correspondence

However, the proof would have to be quite tough as there is no good way to handle such a nonlinear system of elliptic equations. It turns out that Donaldson and Uhlenbeck-Yau were working on this problem independently. I learnt from Hitchin during a trip to England that Donaldson was able to prove the existence for Hermitian connections of any holomorphic vector bundle that can be deformed to the tangent bundle of a K3 surface. (Note that the Ricci-flat metric on a K3 surface provides a natural solution of the Hermitian Yang-Mills connection on the tangent bundle.) This is of course encouraging as it indicates the possibility of the conjecture.

It turns out that Donaldson was concentrated on algebraic surfaces and Uhlenbeck-Yau on arbitrary dimensional Kähler manifolds. While Donaldson used the Bott-Chern form and the Hermitian Yang-Mills flow, Uhlenbeck-Yau constructed a destabilizing sheaf assuming the nonexistence of Hermitian Yang-Mills connection.

The proof of regularity of the destabilizing subsheaf took nontrivial effort and as a result , our paper appeared later than the work of Donaldson's proof for algebraic surfaces. After we published our work, Donaldson found that he can also do the higher dimensional case by restriction of the bundle to hyperplane sections of the algebraic manifold. (It was proved by Maruyama and Mehta-Ramanathan that a stable bundle is stable on a generic hyperplane section.)

This later argument of Donaldson depends intrinsically on the manifold being projective for higher dimensional manifolds, such as the works of Simpson and the works of Bando-Siu, are based on the procedure. However, the later development requires the theorem to be generalized to non-Kähler manifolds. As was acknowledged by Donaldson, the argument of Uhlenbeck-Yau is most natural and in fact, all the later development for Hermitian Yang-Mills connections for higher dimensional manifolds are based on the procedure of Uhlenbeck-Yau.

Some later paper such as the one by Bando-Siu used the Hermitian Yang-Mills flow to generalize our result, but the essential feature of Uhlenbeck-Yau procedure is still needed in an essential manner. It should also be pointed out that the continuity argument used by Uhlenbeck-Yau is just as convenient as the Hermitian Yang-Mills flow.

A few years later, Carlos Simpson generalized the Uhlenbeck-Yau argument to establish similar theorem when the Higgs field was introduced. Hermitian Yang-Mills connections were proposed by me to Edward Witten in 1984 to study heterotic string, which had since became an important subject in mathematical physics. But from the very beginning, we knew the importance of Hermitian Yang-Mills connections.

Balanced metric and Strominger system

Kähler-Einstein metrics are very useful in biregular geometry. We shall discuss it later. However, it cannot answer the important question whether an algebraic manifold is rational or not. The existence of Kähler metric is not a concept that is invariant under birational transformations, while the existence of balanced metric is. The concept of balanced metric was introduced by Michelson. A Hermitian metric is called balanced if its Kähler form satisfied the following equation:

$$d(\omega^{n-1}) = 0$$

and it was proved by Alessandrini and Bassanelli that its existence is invariant under birational transformations. However, there is much more freedom to deform a balanced metric than a Kähler metric. Just demanding that Ricci curvature equal to zero is not enough to determine a unique Balanced metric within the $(n-1, n-1)$ class.

On the other hand, balanced metric comes up naturally in the theory of Heterotic string theory in complex 3-dimension. And (this) balanced condition is related to the concept of supersymmetry. When there is a nonwhere vanishing top dimensional holomorphic 3-form, we look for an Hermitian metric which is balanced, and a stable holomorphic bundle (stable with respect to the balanced metric) whose second Chern form is equal to the second form given by the Hermitian metric.

Altogether, the following equations of the Strominger system need to be satisfied:

(1) $d(\|\Omega\|_\omega \Omega^2) = 0$

(2) $F_h^{2,0} = F_h^{0,2} = 0, \quad F_h \wedge \omega^2 = 0$

(3) $\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{tr}(R_\omega \wedge R_\omega) - \mathrm{tr}(F_h \wedge F_h))$

It provides a natural generalization of the Calabi-Yau geometry, which couples Hermitian metrics with Hermitian Yang-Mills theory. my belief is that the above system of equations can be solved when the obvious conditions hold. Jun Li and I solved this system on any Calab-Yau manifold by making a deformation from the original Calabi-Yau metric.

For some intrinsically non-Kähler manifold, Fu and I solved the Strominger system based on some ansätz for a 3-dimensional complex manifolds obtained from the Calabi-Eckmann construction. (The construction of the non-Kähler manifolds based on Calabi-Eckmann construction was also observed by Goldstein and Prokushkin.) It is a nonsingular complex torus bundle over the K3 surface.

The proof of existence of nonsingular solution to the Strominger system given by Fu-Yau is based on nontrivial estimates related to complex Monge-Ampère equations. In order to understand the significance of Strominger system, Tseng and I, and later with Tsai, developed a new theory of symplectic cohomology which we expect to be dual to this kind of geometry.

Note that the existence of Ricci-flat Kähler metric provides a reduction of holonomic group to a subgroup of $SU(n)$, and according to the work of Candelas-Horowitz-Strominger-Witten, provides a supersymmetric model for vacuum solutions for Type II string theory. They called such manifolds to be Calabi-Yau manifolds. The Strominger system was introduced by Strominger to study Heterotic string where the vacuum is a warped product instead of a direct product.

Questions of Kähler-Einstein metrics in algebraic geometry

There are several interesting consequences of the existence of Kähler-Einstein metric.

A corollary of the above mentioned theorem of Chen-Donaldson-Sun is that the K-stability of such manifold implies that the tangent bundle is stable with respect to the polarization given by the anticanoncial line bundle. This is an interesting statement that is purely algebraic geometric, for which it would be nice to have a proof based only on algebraic geometry.

Also it implies that a K-stable Fano manifold is biregular to $\mathbb{CP}^n$ if the ratio of its two Chern numbers $c_2 c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$.

Understanding of Kähler-Einstein metrics near Singularities

Another interesting question is the following: If a smooth algebraic manifold has Kodaira dimension either equal to the dimension of the manifold or $-\infty$, and if it is minimal in the sense in birational geometry and the ratio of two Chern numbers $c_2 c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$, then the manifold is either $\mathbb{CP}^n$ or complex ball quotient.

For the case of general type, this is likely to be true. But it will be good to allow singular minimal models and in the case of singular algebraic manifolds, we need to define the Chern numbers suitably. This is related to the question of what is the best Kähler-Einstein metric on an algebraic manifold with singularity.

Let us look at the simplest case when the singularity is isolated. If the Kähler metric is complete at the singularity, it is not hard to prove that the Kähler-Einstein metric is unique. However, when it is not complete, it is not necessarily unique. It depends on the behavior of the volume form near the singularity.

What kind of volume forms are allowed? We need to know that the Ricci form of this volume form is positive definite and that the n-fold product of the Ricci form is asymptotic to this volume form near the singularity. (We may require that the metric defined by the Ricci form should have lower bound on its bisectional curvature.)

It would be interesting to classify the asymptotic models of such volume forms. In principe, each of them will give rise to a canonical Kähler-Einstein metric with the given asymptotic behavior of the volume form.

It would be interesting to calculate the contribution of the singularity towards the Chern numbers. An important case is the canonical singularity appearing in the minimal model theory, which we recall below.

Suppose that Y is a normal variety and $f : X \to Y$ be a resolution of the singularities. Then

$$K_X = f^*(K_Y) + \sum_i a_i E_i$$

where the sum is over the irreducible exceptional divisors and the rational numbers a_i are called the discrepancies.

Then the singularities of Y are called canonical if $a_i \geq 0$ for all i and called terminal if $a_i > 0$ for all i.

A 3-dimensional singularity is terminal of index 1 if and only if it is an isolated composite DuVal (cDV) point in $\mathbb{C}^4$. A 3-dimensional terminal singularity of index $r \geq 2$ is a quotient of an isolated cDV point in $\mathbb{C}^4$.

The important question is to find a good Kähler-Einstein metric in a neighborhood of the cDV singularity which is invariant under the group action. For orbifold singularities, one can use those metrics obtained by pushing down from the nonsingular model before quotient by the group.

On the other hand, there may be some other volume form that satisfies the above properties that is distinct from the orbifold construction. The complicated situation is the case that the Ricci form of the volume form may define a metric that is partially going to complete and partially degenerate at the singular point.

It will be important to construct nice model volume form in a neighborhood of the canonical singularities of the manifold whose Ricci form can give rise to a nice metric which is asymptotically Kähler-Einstein.

Kähler-Einstein metrics on quasiprojective varieties and Sasakian-Einstein metrics

In my first paper on the Calabi conjecture, we know that given any Kähler class, we can find a Kähler metric which may degenerate along a divisor whose volume is given by the unique volume defined by the divisor of the pluricanonical sections. How to calculate the second Chern class related to this divisor would be important. The Chern numbers calculated by the degenerate Ricci flat metrics should have residue from the divisor. It would be important to calculate this contribution.

The noncompact version of complete Ricci-flat metric is more complicated, partially because we lack of a good model space to build a good ansatz. At 1978 ICM in Helsinki, I announced the way to build complete noncompact Ricci-flat manifolds.

I conjectured that the manifold can be written as the complement of a divisor D of a compact Kähler manifold M. (It was pointed out by Michael Anderson et al. that we should assume the finiteness of the topology of the manifold, otherwise Taub-NUT manifolds can provide counterexamples.)

My program was to take D to be an anticanonical divisor of M which cannot be contracted to a codimension two subvarity. There will be a holomorphic volume form on M which has poles along D. I expect that this is close enough to provide a necessary and sufficient condition for $M \backslash D$ to admit complete Kähler metric with zero Ricci curvature. When D is nonsingular, I have worked out the program. The details were written up with Tian in two papers.

However, when D has normal crossing singularities, the problem is unsolved, largely because we do not have a good model of complete Ricci-flat metric in a neighborhood of D when D has singularity. An important and interesting case is to allow the complete Kähler metric to have certain type of singularities. Besides quotient singularity, we can allow cone singularity.

In the last case, the interesting examples are metric cones over a Sasakian-Einstein manifold. Important progress was made by Gauntlett, Martelli, Sparks and myself on the existence of Sasakian-Einstein metrics. We gave several obstructions to their existence by studying the Einstein-Hilbert functional restricted to the space of Sasakian-Einstein metrics where it becomes essentially the volume functional. It can further be shown to be a functional of the Reeb vector field associated to the Sasakian structure alone.

We obtain a useful obstruction from the Lichnerowicz bound on the Laplacian which we could identify preicsely as the physics criterion of a unitarity bound in the conformal field theory associated to the hypersurface singularity. We also show that the first variation of the volume functional is related to the Futaki invariant on the Kähler orbifold, hence volume minimization (and a-maximization in the physics language) implies vanishing of Futaki invariant. This includes the cases of regular and quasi-regular Sasakian structures as classified by Reeb vector orbits.

In the irregular case, Collins and Székelyhidi extended the notion of K-semistability to Sasakian structures, showing constant scalar curvature Sasakian metric implies K-semistability and also recovered our results based on the volume functional. The complete classification is still not known, even for complex hypersurfaces with isolated singularity which admits $\mathbb{C}^*$-action.

Thank you!