Conformal Restriction and Brownian Motion

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Contents

Foreword and Summary

The goal of these lectures is to review some of the results related to conformal restriction: the chordal case and the radial case. The audience of the summer school of Mathematical Science Center of Tsinghua University consists senior undergraduates and graduates. Therefore, I assume knowledge in stochastic calculus (Brownian motion, Itô formula etc.) and basic knowledge in complex analysis (Riemann's Mapping Theorem etc.).

These lecture notes are not a compilation of research papers, thus some details in the proofs are omitted. Also partly because of the limited number of lectures, I chose to focus on the main ideas of the proofs. Whereas, I cite the related papers for interested readers.

Of course, I would like to thank my advisor Wendelin Werner with whom I learned the topic on conformal restriction, SLE et al. and solved conformal restriction problem for the radial case. And I want to express my gratitude to all participants of the course, as well as to all colleagues who have sent me their comments and remarks on the previous draft of these notes.

It has been a great pleasure and a rewarding experience to go back to Tsinghua University and to give a lecture here where I spent four years of undergraduate. I owe my thanks to Prof. Yau and Prof. Poon for giving me the chance.

Here is a short description of these notes: In the first introductory lecture, I will briefly describe Brownian intersection exponents and conformal restriction property. The results are collected from [LW99, LW00, LSW01a, LSW01b, LSW02]. In fact, Brownian intersection exponents have close relation with Quantum Field Theory et al. and the interested readers could consult [LW99, DK88] and references there for more background and motivation. The second lecture is a review on Brownian path: Brownian motion, Brownian excursion and Brownian loop. The results are collected from [LW04, Wer05, Wer08, Wer08, SW12, LW04]. The third lecture is an introduction on chordal SLE. Since I only need $\mathrm{SLE}_{8/3}$ in the following of the lecture, I focus on simple SLE paths, i.e. $\kappa \in [0,4]$. For a more complete introduction on SLE, I recommend the readers to read the lecture note by Wendelin Werner [Wer04] or the book by Greg. Lawler [Law05]. The fourth lecture is about the chordal conformal restriction property. The results are collected from [LSW03]. The fifth lecture is an introduction on radial SLE and again, for a more complete introduction on radial SLE, please read [Wer04, Law05]. The sixth lecture is about the radial restriction property. The results are contained in an upcoming paper [Wu13].

In Subsections 4.4 and 6.5, the relation between conformal restriction property and Brownian intersection exponents are discussed. In Subsections 4.5 and 6.6, I gave some related calculation, and I used results from [SW05]. Whereas the calculation is general and the first time readers could skip these two subsections.

1 Lecture 1: Introduction to Brownian intersection exponents and conformal restriction property

1.1 Intersection exponents of Brownian motion

Probabilists and physicists are interested in the property of intersection exponents for two-dimensional Brownian motion (BM for short). Suppose that we have $n+p$ independent planar BMs: $B^1,\cdots,B^n$ and $W^1,\cdots,W^p$. $B^1,\cdots,B^n$ start from the common point $(1,1)$ and $W^1,\cdots,W^p$ start from the common point $(2,1)$. We want to derive the probability that the paths of

$$B^j, j=1,\cdots,n \text{ up to time } t$$

and the paths of

$$W^l, l=1,\cdots,p \text{ up to time } t$$

do not intersect. Precisely,

$$f_{n,p}(t) := \mathbb{P}[\bigcup_{j=1}^{n} B^j[0,t] \bigcap \bigcup_{l=1}^{p} W^l[0,t] = \emptyset].$$

We can see that this probability decays as $t\to\infty$ roughly like a power of t.[1]

The (n,p)-**whole plane intersection exponent** $\xi(n,p)$ is defined by[2]

$$f_{n,p}(t) \approx (\frac{1}{\sqrt{t}})^{\xi(n,p)}, \quad t\to\infty.$$

And we say that $\xi(n,p)$ is the whole plane intersection exponent between one packet of n BMs and one packet of p BMs.

Similarly, we can define more general intersection exponents between $k\geq 2$ packets of BMs containing $p_1,...,p_k$ paths respectively:

$$B^j_l, \quad j=1,...,k, l=1,...,p_j.$$

Each path in jth packet starts from $(j,1)$ and has to avoid all paths of all other packets. The $(p_1,...,p_k)$-whole-plane intersection exponent $\xi(p_1,...,p_k)$ is defined through

$$\mathbb{P}[\bigcup_{u=1}^{p_{j_1}} B^{j_1}_u[0,t] \bigcap \bigcup_{v=1}^{p_{j_2}} B^{j_2}_v[0,t] = \emptyset, 1\leq j_1<j_2\leq k] \approx (\frac{1}{\sqrt{t}})^{\xi(p_1,...,p_k)}, \quad t\to\infty.$$

Another important quantity is **half-plane intersection exponents** of BMs. They are defined exactly as the whole-plane intersection exponents above except that one adds the condition that all BMs (up to time t) remain in the upper half-plane $\mathbb{H} := \{(x,y)\in\mathbb{R}^2 : y>0\}$. We denote these exponents as $\tilde{\xi}(p_1,...,p_k)$. For instance, $\tilde{\xi}(1,1)$ is defined by

$$\mathbb{P}[B[0,t]\bigcap W[0,t] = \emptyset, B[0,t]\subset\mathbb{H}, W[0,t]\subset\mathbb{H}] \approx (\frac{1}{\sqrt{t}})^{\tilde{\xi}(1,1)}, \quad t\to\infty.$$

Physicists have made some striking conjectures about these exponents and they are proved by mathematicians later. We list some of the results here [LW99].

[1] Why? Hint: $f_{n,p}(ts)\approx f_{n,p}(t)f_{n,p}(s)$.

[2] Why $\sqrt{t}$: for BM B, the diameter of $B[0,t]$ scales like $\sqrt{t}$.

1. There is a precise natural meaning of the exponents $\xi(\mu_1,...,\mu_k)$ and $\tilde{\xi}(\lambda_1,...,\lambda_k)$ for positive real numbers $\mu_1,...,\mu_k,\lambda_1,...,\lambda_k$. (See Lecture 4 and Lecture 6).
2. These exponents satisfy certain functional relations
   (a) Cascade relations:
   $$\tilde{\xi}(\lambda_1,...,\lambda_{j-1},\tilde{\xi}(\lambda_j,...,\lambda_k)) = \tilde{\xi}(\lambda_1,...,\lambda_k)$$
   $$\xi(\mu_1,...,\mu_{j-1},\tilde{\xi}(\mu_j,...,\mu_k)) = \xi(\mu_1,...,\mu_k)$$
   (b) Commutation relations:
   $$\tilde{\xi}(\lambda_1,\lambda_2) = \tilde{\xi}(\lambda_2,\lambda_1), \quad \xi(\mu_1,\mu_2) = \xi(\mu_2,\mu_1)$$
3. One can define a positive, strictly increasing continuous function U on $[0,\infty)$ by
   $$U^2(\lambda) = \lim_{N\to\infty} \tilde{\xi}(\underbrace{\lambda,...,\lambda}_{N})/N^2.$$
   Then we have
   $$U(\tilde{\xi}(\lambda_1,...,\lambda_k)) = U(\lambda_1)+\cdots+U(\lambda_k).$$
   This shows in particular that $\tilde{\xi}$ is encoded in U.
4. The whole-plane intersection exponent ξ can be represented as a function of the half-plane intersection exponent $\tilde{\xi}$:
   $$\xi(\mu_1,...,\mu_k) = \eta(\tilde{\xi}(\mu_1,...,\mu_k)).$$
   The function η is called a generalized disconnection exponent and it is a continuous increasing function.
5. Physicists predict that [DK88]
   $$\tilde{\xi}(\underbrace{1,...,1}_{N}) = \frac{1}{3}N(2N+1), \quad \xi(\underbrace{1,...,1}_{N}) = \frac{1}{12}(4N^2-1).$$

Combine all these results, we could predict that (Homework):

$$U(1) = \sqrt{\frac{2}{3}}, \quad U(\lambda) = \frac{\sqrt{24\lambda+1}-1}{\sqrt{24}}$$

$$\tilde{\xi}(\lambda_1,...,\lambda_k) = \frac{1}{24}\left((\sqrt{24\lambda_1+1}+\cdots+\sqrt{24\lambda_k+1}-(k-1))^2-1\right) \tag{1.1}$$

$$\eta(x) = \frac{1}{48}\left((24x+1)^2-4\right)$$

$$\xi(\mu_1,...,\mu_k) = \frac{1}{48}\left((\sqrt{24\mu_1+1}+\cdots+\sqrt{24\mu_k+1}-k)^2-4\right) \tag{1.2}$$

1.2 From Brownian motion to Brownian excursion

Consider the simplest exponent $\tilde{\xi}(1) = 1$. Suppose B is a planar BM started from i, then we have

$$\mathbb{P}[B[0,t] \subset \mathbb{H}] \approx (\frac{1}{\sqrt{t}})^{\tilde{\xi}(1)}.$$

Suppose W is a planar BM started from εi, then

$$\mathbb{P}[W[0,t] \subset \mathbb{H}] \approx (\frac{\varepsilon}{\sqrt{t}})^{\tilde{\xi}(1)},$$

since $(W(\varepsilon^2 t)/\varepsilon, t \geq 0)$ has the same law as B. Consider the law of W conditioned on the event $[W[0,t] \subset \mathbb{H}]$, we can see that the limit as $t \to \infty, \varepsilon \to 0$ exists. We call the limit as Brownian excursion and denote its law as $\mu^\sharp_{\mathbb{H}}(0,\infty)$. There is another equivalent way to define $\mu^\sharp_{\mathbb{H}}(0,\infty)$: Suppose W is a planar BM started from εi, consider the law of W conditioned on the event $[W$ hits $\mathbb{R} + iR$ before $\mathbb{R}]$. Let $R \to \infty, \varepsilon \to 0$, the limit is the same as $\mu^\sharp_{\mathbb{H}}(0,\infty)$. (We will discuss Brownian excursion more in detail in Lecture 2).

Suppose Z is a Brownian excursion, A is a bounded closed subset of $\bar{\mathbb{H}}$ such that $\mathbb{H} \setminus A$ is simply connected and $0 \notin A$. Let Φ_A be the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that

$$\Phi_A(0) = 0, \quad \Phi_A(\infty) = \infty, \quad \Phi_A(z)/z \to 1 \text{ as } z \to \infty.$$

Consider the law of $\Phi_A(Z)$ conditioned on $[Z \cap A = \emptyset]$. We have that, for any function F,

$$\begin{aligned}
&\mathbb{E}[F(\Phi_A(Z)) \,|\, Z \cap A = \emptyset] \\
&= \lim_{R\to\infty,\varepsilon\to 0} \mathbb{E}[F(\Phi_A(W)) \,|\, W \cap A = \emptyset, W \text{ hits } \mathbb{R} + iR \text{ before } \mathbb{R}] \quad \text{(W: BM started from } \varepsilon i) \\
&= \lim_{R\to\infty,\varepsilon\to 0} \mathbb{E}[F(\Phi_A(W)) 1_{W\cap A=\emptyset, W \text{ hits } \mathbb{R}+iR \text{ before } \mathbb{R}}]/\mathbb{P}[W \cap A = \emptyset, W \text{ hits } \mathbb{R} + iR \text{ before } \mathbb{R}] \\
&= \lim_{R\to\infty,\varepsilon\to 0} \mathbb{E}[F(\Phi_A(W)) 1_{W \text{ hits } \mathbb{R}+iR \text{ before } \mathbb{R}} \,|\, W \cap A = \emptyset]/\mathbb{P}[W \text{ hits } \mathbb{R} + iR \text{ before } \mathbb{R} \,|\, W \cap A = \emptyset]
\end{aligned}$$

Conditioned on $[W \cap A = \emptyset]$, the process $\tilde{W} = \Phi_A(W)$ has the same law as a BM started from $\Phi_A(\varepsilon i)$, thus

$$\begin{aligned}
&\mathbb{E}[F(\Phi_A(Z)) \,|\, Z \cap A = \emptyset] \\
&= \lim_{R\to\infty,\varepsilon\to 0} \mathbb{E}[F(\tilde{W}) 1_{\tilde{W} \text{ hits } \Phi_A(\mathbb{R}+iR) \text{ before } \mathbb{R}}]/\mathbb{P}[\tilde{W} \text{ hits } \Phi_A(\mathbb{R} + iR) \text{ before } \mathbb{R}] \\
&= \mathbb{E}[F(Z)].
\end{aligned}$$

In other words, the Brownian excursion Z satisfies the following conformal restriction property: the law of $\Phi_A(Z)$ conditioned on $[Z \cap A = \emptyset]$ is the same as Z itself.

Conformal restriction property is closely related to the half-plane/whole-plane intersection exponents.

1.3 Chordal conformal restriction property

Definition 1.1. *Let $\mathscr{A}_c$ be the collection of all bounded closed subset $A \subset \bar{\mathbb{H}}$ such that*

$$0 \notin A, \quad A = \overline{A \cap \mathbb{H}}, \quad \text{and } \mathbb{H} \setminus A \text{ is simply connected.}$$

Denote Φ_A as the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that[3]

$$\Phi_A(0) = 0, \quad \Phi_A(\infty) = \infty, \quad \Phi_A(z)/z \to 1 \text{ as } z \to \infty.$$

[3] Riemann's Mapping Theorem asserts that, if we have three boundary points, there exists a unique conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that fixes the three points. More detail in Lecture 3.

We are interested in closed random subset K of $\bar{\mathbb{H}}$ such that

1. $K \cap \mathbb{R} = \{0\}$, K is unbounded, K is connected and $\mathbb{H} \setminus K$ has two connected components
2. $\forall \lambda > 0$, λK has the same law as K
3. For any $A \in \mathscr{A}_c$, we have that the law of $\Phi_A(K)$ conditioned on $[K \cap A = \emptyset]$ is the same as K.

The combination of the above properties is called **chordal conformal restriction property**, and the law of such a random set is called **chordal restriction measure**. Clearly, the "fill-in" of the Brownian excursion satisfies the chordal conformal restriction property. And for $n \geq 1$, the "fill-in" of the union of n independent Brownian excursions also satisfies the chordal conformal restriction property.

It turns out that there exists only a one-parameter family $\mathbb{P}(\beta)$ of such probability measures for $\beta \geq 5/8$. (More detail in Lecture 3 and Lecture 4). As expected, for $n \geq 1$, the "fill-in" of the union of n independent Brownian excursions corresponds to $\mathbb{P}(n)$ and, for $\beta \geq 5/8$, $\mathbb{P}(\beta)$ can viewed as the law of a packet of β independent Brownian excursions. The chordal restriction measures are closely related to half-plane intersection exponent (more detail in Lecture 4):

Suppose $K_1, ..., K_p$ are p independent chordal restriction samples of parameters $\beta_1, ..., \beta_p$ respectively. The "fill-in" of the union of these sets

$$\bigcup_{j=1}^{p} K_j$$

conditioned on the event (viewed as a limit)

$$[K_{j_1} \cap K_{j_2} = \emptyset, 1 \leq j_1 < j_2 \leq p]$$

has the same law as a chordal restriction sample of parameter $\tilde{\xi}(\beta_1, ..., \beta_p)$.

1.4 Radial conformal restriction property

Definition 1.2. *Let $\mathscr{A}_r$ be the collection of all compact subset $A \subset \bar{\mathbb{U}}$ such that*

$$0 \notin A, 1 \notin A, \quad A = \overline{A \cap \mathbb{U}}, \quad \text{and } \mathbb{U} \setminus A \text{ is simply connected.}$$

Denote Φ_A as the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ such that[4]

$$\Phi_A(0) = 0, \quad \Phi_A(1) = 1.$$

We are interested in closed random subset K of $\bar{\mathbb{U}}$ such that

1. $K \cap \partial \mathbb{U} = \{1\}$, $0 \in K$, K is connected and $\mathbb{U} \setminus K$ is connected
2. For any $A \in \mathscr{A}_r$, the law of $\Phi_A(K)$ conditioned on $[K \cap A = \emptyset]$ is the same as K.

The combination of the above properties is called **radial conformal restriction property**, and the law of such a random set is called **radial restriction measure**. It turns out there exists only a two-parameter family $\mathbb{Q}(\alpha, \beta)$ of such probability measures for

$$\beta \geq 5/8, \quad \alpha \leq \xi(\beta).$$

[4] Riemann's Mapping Theorem asserts that, if we have one interior point and one boundary point, there exists a unique conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that fixes the interior point and the boundary point.

(More detail in Lecture 5 and Lecture 6). And the radial restriction measures are closely related to whole-plane intersection exponent (more detail in Lecture 6):

Suppose $K_1,...,K_p$ are p independent radial restriction samples of parameters $(\xi(\beta_1),\beta_1),...,(\xi(\beta_p),\beta_p)$ respectively. The "fill-in" of the union of these sets

$$\bigcup_{j=1}^{p} K_j$$

conditioned on the event (viewed as a limit)

$$[K_{j_1}\cap K_{j_2}=\emptyset, 1\leq j_1<j_2\leq p]$$

has the same law as a radial restriction sample of parameter $(\xi(\beta_1,...,\beta_p),\tilde{\xi}(\beta_1,...,\beta_p))$.

1.5 Notations

$$\mathbb{H}=\{z\in\mathbb{C}:\Im(z)>0\}.$$

$$\mathbb{U}=\{z\in\mathbb{C}:|z|<1\},\quad \mathbb{U}(w,r)=\{z\in\mathbb{C}:|z-w|<r\}.$$

Denote

$$f(\varepsilon)\approx g(\varepsilon)\quad\text{as }\varepsilon\to 0,$$

if

$$\lim_{\varepsilon\to 0}\frac{\log f(\varepsilon)}{\log g(\varepsilon)}=1.$$

2 Lecture 2: Brownian motion, excursion and loop

2.1 Brownian motion

1-dimensional BM

Suppose that $(W_t, t \geq 0)$ is a 1-dimensional BM starting from $W_0 = x \in \mathbb{R}$. It is characterized by the following four facts:

1. $W_0 = x$;
2. $(W_t, t \geq 0)$ is a.s. continuous;
3. W has independent increments, i.e. for any $t \geq s \geq 0$, $(W_t - W_s)$ is independent of $(W_u, u \leq s)$;
4. For any $t \geq s \geq 0$, $(W_t - W_s)$ is Gaussian with mean zero and variance $(t - s)$.

We list several basic properties of 1-dimensional BM.

Proposition 2.1. *1. $(W_t, t \geq 0)$ is a martingale.*

2. $(W_t^2 - t, t \geq 0)$ is a martingale. In particular, $\mathbb{E}[W_t^2] = t$.

3. For any $\sigma > 0$, $(\sigma W_{t/\sigma^2}, t \geq 0)$ is a BM.

From the first basic property, we have that the following corollary.

Corollary 2.2. *Suppose that $(W_t, t \geq 0)$ is a BM starting from $W_0 = \varepsilon \in (0,1)$. Define the stopping time*

$$\tau = \inf\{t : W_t = 0 \text{ or } W_t = 1\}.$$

Then

$$\mathbb{P}[W_\tau = 1] = \varepsilon.$$

2-dimensional BM/complex BM

Suppose W^1, W^2 are two independent 1-dimensional BMs, then

$$B = W^1 + iW^2$$

is a complex BM.

Lemma 2.3. *Suppose B is a complex BM and u is a harmonic function, then $u(B)$ is a local martingale.*

Proof. By Itô Formula,

$$du(B_t) = \partial_x u(B_t) dW_t^1 + \partial_y u(B_t) dW_t^2 + \frac{1}{2}(\partial_{xx} u(B_t) + \partial_{yy} u(B_t))dt = \partial_x u(B_t) dW_t^1 + \partial_y u(B_t) dW_t^2.$$

□

Proposition 2.4. *Suppose D is a domain and $f : D \to \mathbb{C}$ is a conformal map. Let B be a complex BM starting from $z \in D$, stopped at*

$$\tau_D := \inf\{t \geq 0 : B_t \notin D\}.$$

Then the time-changed process $f(B)$ has the same law as a complex BM starting from $f(z)$ stopped at $\tau_{f(D)}$. Precisely, define

$$S(t) = \int_0^t |f'(B_u)|^2 du, \quad 0 \leq t < \tau_D,$$

$$\sigma(s) = S^{-1}(s), \quad i.e. \int_0^{\sigma(s)} |f'(B_u)|^2 du = s.$$

Then $(Y_s = f(B_{\sigma(s)}), 0 \leq s \leq S_{\tau_D})$ has the same law as BM starting from $f(z)$ stopped at $\tau_{f(D)}$.

Proof. Write $f = u + iv$ where u, v are harmonic and

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

We have

$$du(B_t) = \partial_x u(B_t) dW_t^1 + \partial_y u(B_t) dW_t^2, \quad dv(B_t) = \partial_x v(B_t) dW_t^1 + \partial_y v(B_t) dW_t^2.$$

Thus the two coordinates of $f(B)$ are local martingales and the quadratic variation is

$$\langle u(B) \rangle_t = \langle v(B) \rangle_t = \int_0^t (\partial_x u^2(B_s) + \partial_y u^2(B_s)) ds = \int_0^t |f'(B_s)|^2 ds,$$

$$\langle u(B), v(B) \rangle_t = (\partial_x u(B_t) \partial_x v(B_t) + \partial_y u(B_t) \partial_y v(B_t)) dt = 0.$$

Thus the two coordinates of Y are independent local martingales with quadratic variation t which implies that Y is a complex BM. □

Notations

Let $\mathcal{K}$ be the set of all parameterized continuous planar curves γ defined on a time interval $[0, t_\gamma]$. $\mathcal{K}$ can be viewed as a metric space

$$d_{\mathcal{K}}(\gamma, \eta) = \inf_{\theta} \sup_{0 \leq s \leq t_\gamma} |s - \theta(s)| + |\gamma(s) - \eta(\theta(s))|$$

where the inf is taken over all increasing homeomorphisms $\theta : [0, t_\gamma] \to [0, t_\eta]$.

If μ is any measure on $\mathcal{K}$, let $|\mu| = \mu(\mathcal{K})$ denote the total mass. If $0 < |\mu| < \infty$, let $\mu^\sharp = \mu / |\mu|$ be μ normalized to be a probability measure. Let M denote the set of finite Borel measures on $\mathcal{K}$. This is a metric space under Prohorov metric [Bil99, Section 6]. To show that a sequence of finite measures μ_n converges to a finite measure μ, it suffices to show that

$$|\mu_n| \to |\mu|, \quad \mu_n^\sharp \to \mu^\sharp.$$

If D is a domain, we say that γ is in D if $\gamma(0, t_\gamma) \subset D$, and let $\mathcal{K}(D)$ be the set of $\gamma \in \mathcal{K}$ that are in D. Note that, we do not require the endpoints of γ to be in D. Suppose $f : D \to D'$ is a conformal map and $\gamma \in \mathcal{K}(D)$. Let

$$S(t) = \int_0^t |f'(\gamma(s))|^2 ds. \tag{2.1}$$

If $S(t) < \infty$ for all $t < t_\gamma$, define $f \circ \gamma$ by

$$(f \circ \gamma)(S(t)) = f(\gamma(t)).$$

If μ is a measure supported on the set of γ in $\mathscr{K}(D)$ such that $f \circ \gamma$ is well-defined and in $\mathscr{K}(D')$, then $f \circ \mu$ denotes the measure

$$f \circ \mu(V) = \mu[\gamma : f \circ \gamma \in V].$$

From interior point to interior point

Let $\mu(z, \cdot; t)$ denote the law of complex BM $(B_s, 0 \le s \le t)$ starting from z. We can write

$$\mu(z, \cdot; t) = \int_{\mathbb{C}} \mu(z, w; t) dA(w)$$

where $dA(w)$ denotes the area measure and $\mu(z, w; t)$ is a measure on continuous curve from z to w. The total mass of $\mu(z, w; t)$ is

$$|\mu(z, w; t)| = \frac{1}{2\pi t} \exp(-\frac{|z-w|^2}{2t}). \tag{2.2}$$

The normalized measure $\mu^\sharp(z, w; t) = \mu(z, w; t)/|\mu(z, w; t)|$ is a probability measure, and it is called a **Brownian bridge** from z to w in time t.

Remark 2.5. $|\mu(z, w; t)|$ *is also called heat kernel and Equation (2.2) can be obtained through*

$$|\mu(z, D; t)| = \mathbb{P}^z[B_t \in D] = \int_D \frac{1}{2\pi t} \exp(-\frac{|z-w|^2}{2t}) dA(w).$$

The measure $\mu(z, w)$ is defined by

$$\mu(z, w) = \int_0^\infty \mu(z, w; t) dt.$$

This is a σ-finite infinite measure. If D is a domain and $z, w \in D$, define $\mu_D(z, w)$ to be $\mu(z, w)$ restricted to curves stayed in D. If $z \neq w$, and D is a domain such a BM in D eventually exits D, then $|\mu_D(z, w)| < \infty$. Define **Green's function**

$$G_D(z, w) = \pi |\mu_D(z, w)|.$$

In particular, $G_{\mathbb{U}}(0, z) = -\log |z|$.

Proposition 2.6. ***(Conformal Invariance)*** *Suppose $f : D \to D'$ is a conformal map, z, w are two interior points in D. Then*

$$f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w)).$$

In particular,

$$G_{f(D)}(f(z), f(w)) = G_D(z, w), \quad (f \circ \mu_D)^\sharp(z, w) = \mu^\sharp_{f(D)}(f(z), f(w)).$$

Homework (important): Prove this proposition. Hint: Proposition 2.4 and Equation (2.1).

From interior point to boundary point

Suppose D is a connected domain. Let B be a BM starting from $z \in D$ and stopped at

$$\tau_D = \inf\{t : B_t \notin D\}.$$

Define $\mu_D(z, \partial D)$ to be the law of $(B_s, 0 \le s \le \tau_D)$. If D has nice boundary (i.e. ∂D is piecewise analytic), we can write

$$\mu_D(z, \partial D) = \int_{\partial D} \mu_D(z, w) dw$$

where dw is the length measure and $\mu_D(z, w)$ is a measure on continuous curves from z to w. Define **Poisson's kernel**

$$H_D(z, w) = |\mu_D(z, w)|.$$

In particular, $H_{\mathbb{U}}(0, w) = 1/(2\pi)$.

The normalized measure $\mu_D^\sharp(z, w) = \mu_D(z, w)/|\mu_D(z, w)|$ can also be viewed as the law of BM conditioned to exit D at w when w is a nice boundary point:

$$\begin{aligned} & \mathbb{P}^z[\cdot \,|\, B_{\tau_D} \in \mathbb{U}(w, \varepsilon)] \\ = \; & \mu_D(z, \mathbb{U}(w, \varepsilon))[\cdot]/|\mu_D(z, \mathbb{U}(w, \varepsilon))| \\ = \; & \int_{\mathbb{U}(w,\varepsilon)} \mu_D(z, u)[\cdot] du \Big/ \int_{\mathbb{U}(w,\varepsilon)} |\mu_D(z, u)| du \to \mu_D^\sharp(z, w) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Proposition 2.7. *(**Conformal Covariance**) Suppose D is a connected domain with nice boundary, $z \in D, w \in \partial D$ is a nice boundary point. Let $f : D \to D'$ be a conformal map. Then*

$$f \circ \mu_D(z, w) = |f'(w)| \mu_{f(D)}(f(z), f(w)).$$

In particular,

$$|f'(w)| H_{f(D)}(f(z), f(w)) = H_D(z, w), \quad (f \circ \mu_D)^\sharp(z, w) = \mu^\sharp_{f(D)}(f(z), f(w)).$$

Relation between the two

Proposition 2.8. *Suppose D is a connected domain with nice boundary, $z \in D, w \in \partial D$ is a nice boundary point. Let $\mathbf{n}_w$ denote the inward normal at w, then*

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mu_D(z, w + \varepsilon \mathbf{n}_w) = \mu_D(z, w).$$

In particular,

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi\varepsilon} G_D(z, w + \varepsilon \mathbf{n}_w) = H_D(z, w), \quad \mu_D^\sharp(z, w_n) \to \mu_D^\sharp(z, w) \text{ as } w_n \in D \to w.$$

2.2 Brownian excursion

Suppose D is a connected domain with nice boundary and z, w are two distinct nice boundary points. Define the measure on Brownian path from z to w in D:

$$\mu_D(z, w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu_D(z + \varepsilon \mathbf{n}_z, w) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2} \mu_D(z + \varepsilon \mathbf{n}_z, w + \varepsilon \mathbf{n}_w).$$

Denote

$$H_D(z,w) = |\mu_D(z,w)|.$$

The normalized measure $\mu_D^\sharp(z,w)$ is called Brownian excursion measure in D with two end points $z,w \in \partial D$. Note that

$$H_D(z,w) = \lim_{\varepsilon \to 0} H_D(z+\varepsilon \mathbf{n}_z, w), \quad H_{\mathbb{H}}(0,x) = \frac{1}{\pi x^2}.$$

Proposition 2.9. *(**Conformal Covariance**) Suppose that $f : D \to D'$ is a conformal map, and $z,w \in \partial D$, $f(z), f(w) \in \partial f(D)$ are nice boundary points. Then*

$$f \circ \mu_D(z,w) = |f'(z)f'(w)|\mu_{f(D)}(f(z),f(w)).$$

In particular,

$$|f'(z)f'(w)|H_{f(D)}(f(z),f(w)) = H_D(z,w), \quad (f \circ \mu_D)^\sharp(z,w) = \mu_{f(D)}^\sharp(f(z),f(w)).$$

The following proposition is an equivalent expression of the conformal restriction property of Brownian excursion we discussed in Subsection 1.2.

Proposition 2.10. *Suppose $A \in \mathscr{A}_c$ and Φ_A is the conformal map defined in Definition 1.1. Let e be a Brownian excursion whose law is $\mu_{\mathbb{H}}^\sharp(0,\infty)$. Then*

$$\mathbb{P}[e \cap A = \emptyset] = \Phi_A'(0).$$

Proof. Although $\mu_{\mathbb{H}}(0,\infty)$ has zero total mass, the normalized measure can still be defined through the limit procedure:

$$\mu_{\mathbb{H}}^\sharp(0,\infty) = \lim_{x\to\infty} \mu_{\mathbb{H}}^\sharp(0,x) = \lim_{x\to\infty} \mu_{\mathbb{H}}(0,x)/|\mu_{\mathbb{H}}(0,x)|.$$

Thus

$$\begin{aligned}
&\mathbb{P}[e \cap A = \emptyset] \\
&= \lim_{x\to\infty} \mu_{\mathbb{H}}(0,x)[e \cap A = \emptyset]/|\mu_{\mathbb{H}}(0,x)| \\
&= \lim_{x\to\infty} |\mu_{\mathbb{H}\setminus A}(0,x)|/|\mu_{\mathbb{H}}(0,x)| \\
&= \lim_{x\to\infty} H_{\mathbb{H}\setminus A}(0,x)/H_{\mathbb{H}}(0,x) \\
&= \lim_{x\to\infty} \Phi_A'(0)\Phi_A'(x)H_{\mathbb{H}}(0,\Phi_A(x))/H_{\mathbb{H}}(0,x) = \Phi_A'(0).
\end{aligned}$$

□

Corollary 2.11. *Suppose $e_1,\ldots,e_n$ are n independent Brownian excursion with law $\mu_{\mathbb{H}}^\sharp(0,\infty)$, denote $\Sigma = \cup_{j=1}^n e_j$, then for any $A \in \mathscr{A}_c$,*

$$\mathbb{P}[\Sigma \cap A = \emptyset] = \Phi_A'(0)^n.$$

Corollary 2.12. *Let e be a Brownian excursion with law $\mu_{\mathbb{H}}^\sharp(x,y)$ where $x,y \in \mathbb{R}, x \neq y$. Then, for any closed subset $A \subset \bar{\mathbb{H}}$ such that $x,y \notin A$ and $\mathbb{H}\setminus A$ is simply connected, we have that*

$$\mathbb{P}[e \cap A = \emptyset] = \Phi'(x)\Phi'(y)$$

where Φ is any conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that fixes x and y.

Homework: Prove this corollary and check that the quantity $\Phi'(x)\Phi'(y)$ is unique although Φ is not unique.

Definition 2.13. *Suppose D has nice boundary, then* ***Brownian excursion measure*** *is defined as*

$$\mu^{exc}_{D,\partial D} = \int_{\partial D}\int_{\partial D} \mu_D(z,w)dzdw.$$

Generally, if I is a subsect of ∂D, define

$$\mu^{exc}_{D,I} = \int_I\int_I \mu_D(z,w)dzdw.$$

Proposition 2.14. *(**Conformal Invariance**) Suppose D, D' have nice boundaries and $f : D \to D'$ is a conformal map. Then*

$$f\circ\mu^{exc}_{D,I} = \mu^{exc}_{f(D),f(I)}, \quad f\circ\mu^{exc}_{D,\partial D} = \mu^{exc}_{f(D),\partial f(D)}.$$

Proof.

$$\begin{aligned} & f\circ\mu^{exc}_{D,I} \\ =\ & \int_I\int_I f\circ\mu_D(z,w)dzdw \\ =\ & \int_I\int_I |f'(z)f'(w)|\mu_{f(D)}(f(z),f(w))dzdw \\ =\ & \int_{f(I)}\int_{f(I)} \mu_{f(D)}(z,w)dzdw = \mu^{exc}_{f(D),f(I)}. \end{aligned}$$

□

Theorem 2.15. *Let $(e_j, j\in J)$ be a Poisson point process with intensity $\pi\beta\mu^{exc}_{\mathbb{H},\mathbb{R}_-}$ for some $\beta > 0$. Set $\Sigma = \cup_j e_j$. For any $A \in \mathscr{A}_c$ such that $A\cap\mathbb{R} \subset (0,\infty)$, we have that*

$$\mathbb{P}[\Sigma\cap A = \emptyset] = \Phi'_A(0)^\beta.$$

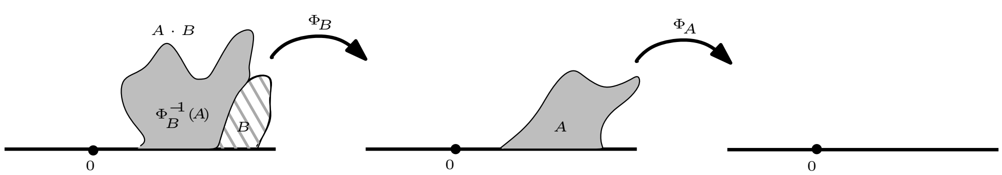


Fig. 2.1: $A\cdot B$ is the union of $\Phi_B^{-1}(A)$ and B.

Proof. Since

$$\mathbb{P}[\Sigma\cap A = \emptyset] = \exp(-\pi\beta\mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A \neq \emptyset]),$$

we need to show that

$$\mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\neq\emptyset] = -\frac{1}{\pi}\log\Phi'_A(0).$$

This will be obtained by two steps: First, there exists a constant c such that

$$\mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\neq\emptyset] = c\log\Phi'_A(0). \tag{2.3}$$

Second,

$$c = -1/\pi. \tag{2.4}$$

For the first step, we need to introduce a set $A\cdot B$: Suppose $A,B \in \mathscr{A}_c$ such that $A\cap\mathbb{R}\subset(0,\infty)$ and $B\cap\mathbb{R}\subset(0,\infty)$. Define (see Figure 2.1)

$$A\cdot B = \Phi_B^{-1}(A)\cap B.$$

Then clearly, $\Phi_{A\cdot B} = \Phi_A\circ\Phi_B$, and

$$\log\Phi'_{A\cdot B}(0) = \log\Phi'_A(0) + \log\Phi'_B(0). \tag{2.5}$$

For the Brownian excursion measure, we have

$$\begin{aligned}
&\mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\cdot B\neq\emptyset]\\
=\;& \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap B\neq\emptyset] + \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap B=\emptyset, e\cap A\cdot B\neq\emptyset]\\
=\;& \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap B\neq\emptyset] + \mu^{exc}_{\mathbb{H}\setminus B,\mathbb{R}_-}[e\cap \Phi_B^{-1}(A)\neq\emptyset]\\
=\;& \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap B\neq\emptyset] + \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\neq\emptyset]
\end{aligned}$$

In short, we have

$$\mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\cdot B\neq\emptyset] = \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap B\neq\emptyset] + \mu^{exc}_{\mathbb{H},\mathbb{R}_-}[e\cap A\neq\emptyset].$$

Compared with Equation (2.5), we have Equation (2.3).[5] Generally, if $I=[a,b]\subset\mathbb{R}_-$, we have

$$\mu^{exc}_{\mathbb{H},I}[e\cap A\neq\emptyset] = c\log(\Phi'_A(a)\Phi'_A(b)). \tag{2.6}$$

Next, we will decide the constant. Suppose $I=[a,b]\subset\mathbb{R}_-$ and $A\in\mathscr{A}_c$ such that $A\cap\mathbb{R}\subset(0,\infty)$.

$$\begin{aligned}
&\mu^{exc}_{\mathbb{H},I}[e\cap A\neq\emptyset]\\
=\;& \int_I\int_I \mu_{\mathbb{H}}(x,y)[e\cap A\neq\emptyset]dxdy\\
=\;& \int_I\int_I H_{\mathbb{H}}(x,y)\mu^{\sharp}_{\mathbb{H}}(x,y)[e\cap A\neq\emptyset]dxdy\\
=\;& \int_I\int_I H_{\mathbb{H}}(x,y)(1-\Phi'_{x,y}(x)\Phi'_{x,y}(y))dxdy \quad (\textit{Corollary } 2.12)
\end{aligned}$$

where $\Phi_{x,y}$ is any conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that fixes x and y. Define the Mobius transformation

$$m(z) = \left(\frac{x-y}{\Phi_A(x)-\Phi_A(y)}\right)(z-x)+x,$$

then $\Phi_{x,y} = m\circ\Phi_A$ would do the work. Thus

$$\mu^{exc}_{\mathbb{H},I}[e\cap A\neq\emptyset] = \int_I\int_I \frac{1}{\pi|x-y|^2}\left(1-\left(\frac{x-y}{\Phi_A(x)-\Phi_A(y)}\right)^2\Phi'_A(x)\Phi'_A(y)\right)dxdy.$$

It is not clear to see how this double integral would give $c\log(\Phi'_A(a)\Phi'_A(b))$. However, we only need to decide the the constant c which is much easier. Suppose $I=[-\varepsilon,0]$, and set $a_1=\Phi'_A(0)$ and $a_2=\Phi''_A(0)/2$, we have that

$$\mu^{exc}_{\mathbb{H},I}[e\cap A\neq\emptyset] = \frac{a_2^2}{\pi a_1^2}\varepsilon^2 + o(\varepsilon^2),$$

$$\log(\Phi'_A(0)\Phi'_A(-\varepsilon)) = -\frac{a_2^2}{a_1^2}\varepsilon^2 + o(\varepsilon^2).$$

Compare the two expansions, the constant $c=-1/\pi$. □

[5] Idea: $F(t+s)=F(t)+F(s)\rightsquigarrow F(t)=ct$. For precise proof, see [Wer05, Theorem 8].

2.3 Brownian loop

Suppose $(\gamma(t), 0 \le t \le t_\gamma) \in \mathscr{K}$ is a loop, i.e. $\gamma(0) = \gamma(t_\gamma)$. Such a γ can be considered as a function defined on $(-\infty, \infty)$ satisfying $\gamma(s) = \gamma(s + t_\gamma)$ for any $s \in \mathbb{R}$. Let $\tilde{\mathscr{K}} \subset \mathscr{K}$ be the collection of such loops. Define, for $r \in \mathbb{R}$, the shift operator θ_r on loops:

$$\theta_r \gamma(s) = \gamma(r+s).$$

We say that two loops γ, γ' are equivalent if for some r, we have $\gamma' = \theta_r \gamma$. Denote $\tilde{\mathscr{K}}_u$ as the set of unrooted loops, i.e. the equivalent classes. We will define Brownian loop measure on unrooted loops.

Recall that $\mu(z, \cdot; t)$ denotes the law of complex BM $(B_s, 0 \le s \le t)$ and

$$\mu(z, \cdot; t) = \int \mu(z, w; t) dA(w).$$

Now we are interested in loops, i.e. $\mu(z, z; t)$ where the path starts from z and returns back to z. We have that

$$|\mu(z, z; t)| = \frac{1}{2\pi t}, \quad \mu(z, z) = \int_0^\infty \mu(z, z; t) dt = \int_0^\infty \frac{1}{2\pi t} \mu^\sharp(z, z; t) dt.$$

We define **Brownian loop measure** μ^{loop} by

$$\mu^{loop} = \int_{\mathbb{C}} \frac{1}{t_\gamma} \mu(z, z) dA(z) = \int_{\mathbb{C}} \int_0^\infty \frac{1}{2\pi t^2} \mu^\sharp(z, z; t) dt dA(z). \tag{2.7}$$

The term $1/t_\gamma$ corresponds to averaging over the root and μ^{loop} is defined on unrooted loops.

If D is a domain, define μ_D^{loop} to be μ^{loop} restricted to the curves totally contained in D.

Proposition 2.16. ***(Conformal Invariance)*** *If $f : D \to D'$ is a conformal map, then*

$$f \circ \mu_D^{loop} = \mu_{f(D)}^{loop}.$$

Proof. We call a Borel measurable function $T : \tilde{\mathscr{K}} \to [0, \infty)$ a unit weight if, for any $\gamma \in \tilde{\mathscr{K}}$, we have

$$\int_0^{t_\gamma} T(\theta_r \gamma) = 1.$$

One example is $T(\gamma) = 1/t_\gamma$. For any unit weight T, since μ^{loop} is defined on unrooted loops, we have that

$$\mu^{loop} = \int_{\mathbb{C}} T \mu(z, z) dA(z). \tag{2.8}$$

We will give a short proof of Equation (2.8). For any function G on $\tilde{\mathscr{K}}$, it induces a function G_u on $\tilde{\mathscr{K}}_u$:

$$G_u([\gamma]) = \frac{1}{t_\gamma} \int_0^{t_\gamma} G(\theta_r \gamma) dr.$$

For any function F on $\tilde{\mathscr{K}}_u$, we define

$$G^1(\gamma) = F([\gamma]), \quad G^2(\gamma) = t_\gamma T F([\gamma]).$$

It is easy to see that $G_u^1 = G_u^2 = F$. Thus

$$\mu^{loop}[F] = \int \frac{1}{t_\gamma} \mu(z, z)[G^1] dA(z) = \int \frac{1}{t_\gamma} \mu(z, z)[F] dA(z),$$

and

$$\mu^{loop}[F] = \int \frac{1}{t_\gamma}\mu(z,z)[G^2]dA(z) = \int T(\gamma)\mu(z,z)[F]dA(z).$$

Compare the two, we get Equation (2.8).

Now we are ready to prove the proposition. Define T_f: for any $\gamma \in \tilde{\mathscr{K}}$,

$$T_f(\gamma) = |f'(\gamma(0))|^2/t_{f\circ\gamma}.$$

Recall the time change in Equation (2.1), we can see that T_f is a unit weight:

$$\int_0^{t_\gamma} T_f(\theta_r\gamma)dr = \int_0^{t_\gamma} |f'(\gamma(r))|^2/t_{f\circ\gamma}dr = 1.$$

Thus,

$$\mu_D^{loop} = \int_D T_f\mu_D(z,z)dA(z) = \int_D \frac{1}{t_{f\circ\gamma}}|f'(z)|^2\mu_D(z,z)dA(z).$$

$$\begin{aligned} & f\circ\mu_D^{loop} \\ = & \int_D \frac{1}{t_{f\circ\gamma}}|f'(z)|^2 f\circ\mu_D(z,z)dA(z) \\ = & \int_D \frac{1}{t_{f\circ\gamma}}|f'(z)|^2\mu_{f(D)}(f(z),f(z))dA(z) \\ = & \int_{f(D)} \frac{1}{t_\eta}\mu_{f(D)}(w,w)dA(w) = \mu_{f(D)}^{loop}. \end{aligned}$$

□

Denote $\mu_{\mathbb{U},0}^{loop}$ as $\mu_{\mathbb{U}}^{loop}$ restricted to the loops surrounding the origin.

Theorem 2.17. *Let $(l_j, j\in J)$ be a Poisson point process with intensity $\alpha\mu_{\mathbb{U},0}^{loop}$ for some $\alpha > 0$. Set $\Sigma = \cup_j l_j$. For any closed subset $A\subset\bar{\mathbb{U}}$ such that $0\notin A$, $\mathbb{U}\setminus A$ is simply connected, we have that*

$$\mathbb{P}[\Sigma\cap A = \emptyset] = \Phi_A'(0)^{-\alpha}$$

where Φ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ with $\Phi_A(0) = 0, \Phi_A'(0) > 0$.

Proof. Since

$$\mathbb{P}[\Sigma\cap A = \emptyset] = \exp(-\alpha\mu_{\mathbb{U},0}^{loop}(l\cap A\neq\emptyset)),$$

we need to show that

$$\mu_{\mathbb{U},0}^{loop}[l\cap A\neq\emptyset] = \log\Phi_A'(0).$$

The same as before, this can be obtained by two steps: First, there exists a constant c such that

$$\mu_{\mathbb{U},0}^{loop}[l\cap A\neq\emptyset] = c\log\Phi_A'(0). \tag{2.9}$$

Second,

$$c = 1. \tag{2.10}$$

For the first step, it can be proved in the similar way as the proof of the first step of Theorem 2.15, and the precise proof can be found in [Wer08, Lemma 4]. But for the second step, it is more complicate. We omit this part and the interested readers can consult [Wer08, SW12, LW04]. □

3 Lecture 3: Chordal SLE

3.1 Introduction

Schramm Lowner Evolution (SLE for short) is introduced by Oded Schramm in 1999 [Sch00] as the candidates of the scaling limits of discrete statistical physics models. We will take percolation as an example. Suppose D is a domain and we have a discrete lattice of size ε inside D, say the triangular lattice $\varepsilon\mathbb{T}\cap D$. The critical percolation on the discrete lattice is the following: At each vertex of the lattice, there is a random variable which is black or white with equal probability $1/2$. All these random variables are independent. We can see that there are interfaces separating black vertices from white vertices. To be precise, let us fix two distinct boundary points $a,b\in\partial D$. Denote ∂_L (resp. ∂_R) as the part of the boundary from a to b clockwise (resp. counterclockwise). We fix all vertices on ∂_L (resp. ∂_R) to be white (resp. black). And then sample independent black/white random variables at the vertices inside D. Then there exists a unique interface from a to b separating black vertices from white vertices (see Figure 3.1). We denote this interface as γ^ε, and call it as the critical percolation interface in D from a to b.

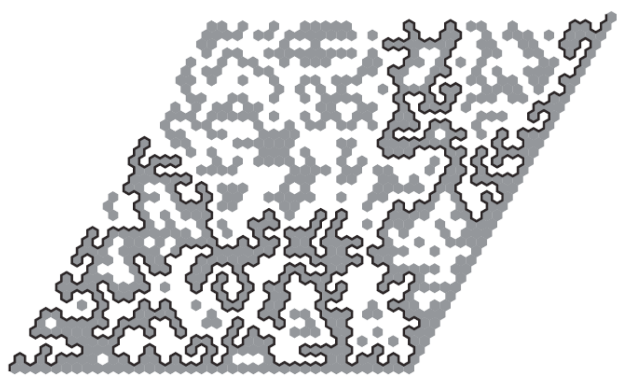
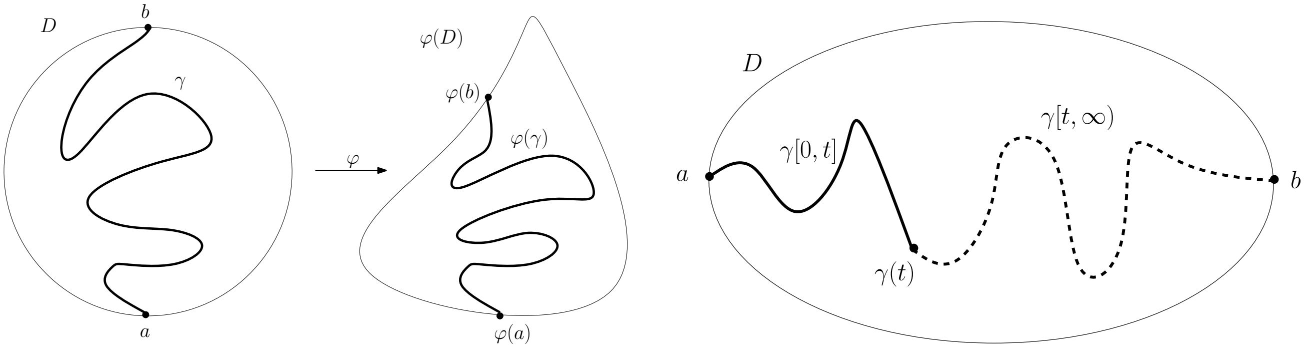


Fig. 3.1: There exists a unique interface from the left-bottom corner to right-top corner separating black vertices from white vertices. (Picture by Julien Dubédat, from [Wer07])

It is worthwhile to point out the domain Markov property in this discrete model: Starting from a, we move along γ^ε and stopped at some point $\gamma^\varepsilon(n)$. Given $L=(\gamma^\varepsilon(1),...,\gamma^\varepsilon(n))$, the future part of γ^ε has the same law as the critical percolation interface in $D\setminus L$ from $\gamma^\varepsilon(n)$ to b.

People believe that the discrete interface γ^ε will converge to some continuous path in D from a to b as ε goes to zero. Assume this is true and suppose γ is the limit continuous curve in D from a to b. Then we would expect that the limit should satisfies the following two properties: Conformal Invariance and Domain Markov Property which is the continuous analog of discrete domain Markov property. SLE curves are introduced from this motivation: chordal SLE curves are random curves in simply connected domains connecting two boundary points such that they satisfy: (see Figure 3.2)

- **Conformal Invariance**: γ is an SLE curve in D from a to b, φ is a conformal map, then $\varphi(\gamma)$ has the same law as an SLE curve in $\varphi(D)$ from $\varphi(a)$ to $\varphi(b)$.
- **Domain Markov Property**: γ is an SLE curve in D from a to b, given $\gamma([0,t])$, $\gamma([t,\infty))$ has the same law as an SLE curve in $D\setminus\gamma[0,t]$ from $\gamma(t)$ to b.



(a) Conformal Invariance. (b) Domain Markov Property.

Fig. 3.2: Characterization of SLE.

The following of the lecture is organized as follows: In Subsection 3.2, we introduce one time parameterization of continuous curves, called Loewner chain, that is suitable to describe the domain Markov property of the curves. In Subsection 3.3, we introduce the definition of chordal SLE and discuss its basic properties.

Without loss of generality, we choose to work in the upper half-plane $\mathbb{H}$ and suppose the two boundary points are 0 and ∞.

3.2 Loewner chain

Half-plane capacity

We call a compact subset K of $\bar{\mathbb{H}}$ a hull if $H = \mathbb{H} \setminus K$ is simply connected. Riemann's mapping theorem asserts that there exists a conformal map Ψ from H onto $\mathbb{H}$ that $\Psi(\infty) = \infty$. In fact, if Ψ is such a map, then $c\Psi + c'$ for $c > 0, c' \in \mathbb{R}$ is also a map from H onto ∞ fixing ∞. We choose to fix the two-degree freedom in the following way. Since Ψ can be expanded near ∞: there exist $b_1, b_0, b_{-1}, \dots$

$$\Psi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \cdots + \frac{b_{-n}}{z^n} + o(z^{-n}), \quad \text{as } z \to \infty.$$

Furthermore, since Ψ preserves the real axis near ∞, all coefficients b_j are real. Hence, for each K, there exists a unique conformal map Ψ from $H = \mathbb{H} \setminus K$ onto $\mathbb{H}$ such that

$$\Psi(z) = z + 0 + O(1/z), \quad \text{as } z \to \infty.$$

We call such a conformal map as the conformal map from $H = \mathbb{H} \setminus K$ onto $\mathbb{H}$ normalized at ∞, and denote it as Ψ_K. In particular, there exists a real $a = a(K)$ such that

$$\Psi(z) = z + \frac{2a}{z} + o(\frac{1}{z}), \quad \text{as } z \to \infty.$$

This number $a(K)$ is a way to measure the size of K.

Lemma 3.1. *$a(K)$ is a non-negative increasing function of the set K.*

Proof. We first show that a is non-negative. Suppose that $Z = X + iY$ is a complex BM starting from $Z_0 = iy$ for some $y > 0$ large (so that $iy \in H = \mathbb{H} \setminus K$) and stopped at its first exit time τ of H. Let Ψ be the conformal map from H onto $\mathbb{H}$ normalized at the infinity, then $\Im(\Psi(z)-z)$ is a bounded harmonic function in H. The martingale stopping theorem therefore shows that

$$\mathbb{E}[\Im(\Psi(Z_\tau))-Y_\tau]=\Im(\Psi(iy)-iy)=-\frac{2a}{y}+o(\frac{1}{y}),\quad \text{as } y\to\infty.$$

Since $\Psi(Z_\tau)$ is real, we have that

$$2a=\lim_{y\to\infty} y\mathbb{E}[Y_\tau]\geq 0.$$

Next we show that a is increasing. Suppose K,K' are hulls and $K\subset K'$. Let $\Psi_1=\Psi_K$, and let Ψ_2 be the conformal map from $\mathbb{H}\setminus\Psi_K(K'\setminus K)$ onto $\mathbb{H}$ normalized at infinity. Then $\Psi_{K'}=\Psi_2\circ\Psi_1$, and

$$a(K')=a(K)+a(\Psi_2)\geq a(K).$$

□

We call $a(K)$ as the capacity of K in $\mathbb{H}$ seen from ∞ or **half-plane capacity**. Here are several simple facts

- When K is vertical slit $[0,iy]$, we have $\Psi_K(z)=\sqrt{z^2+y^2}$. In particular, $a(K)=y^2/4$.
- If $\lambda>0$, then $a(\lambda K)=\lambda^2 a(K)$

Homework: check these two items.

Loewner chain

Suppose that $(W_t,t\geq 0)$ is a continuous real function with $W_0=0$. For each $z\in\bar{\mathbb{H}}$, define the function $g_t(z)$ as the solution to the ODE

$$\partial_t g_t(z)=\frac{2}{g_t(z)-W_t},\quad g_0(z)=z.$$

This is well-defined as long as $g_t(z)-W_t$ does not hit 0. Define

$$T(z)=\sup\{t>0:\min_{s\in[0,t]}|g_s(z)-W_s|>0\}.$$

This is the largest time up to which $g_t(z)$ is well-defined. Set

$$K_t=\{z\in\bar{\mathbb{H}}:T(z)\leq t\},\quad H_t=\mathbb{H}\setminus K_t.$$

We can check that

- g_t is a conformal map from H_t onto $\mathbb{H}$ normalized at ∞.
- For each t,
$$g_t(z)=z+\frac{2t}{z}+o(\frac{1}{z}),\quad \text{as } z\to\infty.$$
In other words, $a(K_t)=t$.

The family $(K_t,t\geq 0)$ is called the **Loewner chain** driven by $(W_t,t\geq 0)$.
One example: $W_t=0$. Then $g_t(z)=\sqrt{z^2+4t}$ and $K_t=[0,2i\sqrt{t}]$.

3.3 Chordal SLE

Definition

Chordal SLE$_\kappa$ for $\kappa \geq 0$ is the Loewner chain driven by $W_t = \sqrt{\kappa} B_t$ where B is a 1-dimensional BM starting from 0.

Lemma 3.2. *1. Chordal SLE$_\kappa$ is scale-invariant.*

2. Chordal SLE$_\kappa$ satisfies domain Markov property.

3. The law of SLE$_\kappa$ is symmetric with respect to the imaginary axis.

Proof. Proof of scale-invariance: Since W is scale-invariant, i.e. for any $\lambda > 0$, the process $W_t^\lambda = W_{\lambda t}/\sqrt{\lambda}$ has the same law as W. Set $g_t^\lambda(z) = g_{\lambda t}(\sqrt{\lambda} z)/\sqrt{\lambda}$, we have

$$\partial_t g_t^\lambda(z) = \frac{2}{g_t^\lambda - W_t^\lambda}, \quad g_0^\lambda(z) = z.$$

Thus $(K_{\lambda t}/\sqrt{\lambda}, t \geq 0)$ has the same law as K.

Proof of domain Markov property: Since BM is a stong Markov process with independent increments. Thus for any stopping time T, the process $(g_T(K_{t+T} \setminus K_T) - W_T, t \geq 0)$ is independent of $(K_s, 0 \leq s \leq T)$ and has the same law as K.

Proof of symmetry: W and $-W$ has the same law. □

Proposition 3.3. *For all $\kappa \in [0,4]$, chordal SLE$_\kappa$ is almost surely generated by a simple continuous curve, i.e. there exists a simple continuous curve γ such that $K_t = \gamma[0,t]$ for all $t \geq 0$. See Figure 3.3. And almost surely,*

$$\lim_{t\to\infty} \gamma(t) = \infty.$$

The proof of this proposition is difficult, we will omit it in the lecture. The interested readers could consult [RS05].

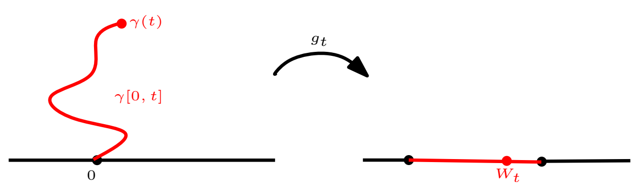


Fig. 3.3: g_t is the conformal map from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ normalized at ∞. And the tip of the curve $\gamma(t)$ is the preimage of W_t under g_t: $\gamma(t) = g_t^{-1}(W_t)$.

Restriction property of SLE$_{8/3}$

In this part, we will compute the probability of SLE$_{8/3}$ γ to avoid a set $A \in \mathscr{A}_c$. To this end, we need to analyze the behavior of the image $\tilde{\gamma} = \Phi_A(\gamma)$:

$$T = \inf\{t : \gamma(t) \in A\}. \quad \text{For } t < T,\ \tilde{\gamma}[0,t] := \Phi_A(\gamma[0,t]).$$

Recall that Φ_A is the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $\Phi_A(0) = 0$, $\Phi_A(\infty) = \infty$, and $\Phi_A(z)/z \to 1$ as $z \to \infty$. g_t is the conformal map from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ normalized at infinity. Define $\tilde{g}_t$ as the conformal map from $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ onto $\mathbb{H}$ normalized at infinity and h_t as the conformal map from $\mathbb{H} \setminus g_t(A)$ onto $\mathbb{H}$ such that Equation (3.1) holds. See Figure 3.4.

$$h_t \circ g_t = \tilde{g}_t \circ \Phi_A. \tag{3.1}$$

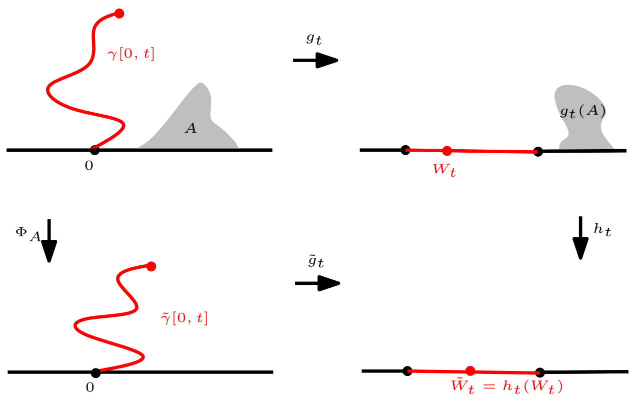


Fig. 3.4: Φ_A is the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $\Phi_A(0) = 0$, $\Phi_A(\infty) = \infty$, and $\Phi_A(z)/z \to 1$ as $z \to \infty$. g_t is the conformal map from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ normalized at infinity. $\tilde{g}_t$ is the conformal map from $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ onto $\mathbb{H}$ normalized at infinity. h_t is the conformal map from $\mathbb{H} \setminus g_t(A)$ onto $\mathbb{H}$ such that $h_t \circ g_t = \tilde{g}_t \circ \Phi_A$.

Proposition 3.4. *When $\kappa = 8/3$, the process*

$$M_t = h_t'(W_t)^{5/8}, \quad t < T$$

is a local martingale.

Proof. Define

$$a(t) = a(\gamma[0,t] \cap A) = a(A) + a(\tilde{\gamma}[0,t]).$$

A time-change shows that

$$\partial_t \tilde{g}_t(z) = \frac{2\partial_t a}{\tilde{g}_t(z) - \tilde{W}_t}.$$

Plugin Equation (3.1), we have that

$$(\partial_t h_t)(z) + h_t'(z)\frac{2}{z - W_t} = \frac{2\partial_t a}{h_t(z) - h_t(W_t)}. \tag{3.2}$$

We can first decide $\partial_t a$: multiply $h_t(z) - h_t(W_t)$ to both sides of Equation (3.2), and then let $z \to W_t$, we have

$$\partial_t a = h_t'(W_t)^2.$$

Then Equation (3.2) becomes

$$(\partial_t h_t)(z) = \frac{2h_t'(W_t)^2}{h_t(z) - h_t(W_t)} - \frac{2h_t'(z)}{z - W_t}. \tag{3.3}$$

Differentiate Equation (3.3) with respect to z, we have

$$(\partial_t h_t)'(z) = \frac{-2h_t'(W_t)^2 h_t'(z)}{(h_t(z)-h_t(W_t))^2} + \frac{2h_t'(z)}{(z-W_t)^2} - \frac{2h_t''(z)}{z-W_t}.$$

Let $z \to W_t$, we have

$$(\partial_t h_t')(W_t) = h_t''(W_t)dW_t + \left(\frac{h_t''(W_t)^2}{2h_t'(W_t)} + (\frac{\kappa}{2} - \frac{4}{3})h_t'''(W_t)\right)dt.$$

When $\kappa = 8/3$,

$$d(h_t'(W_t))^{5/8} = \frac{5h_t''(W_t)}{8h_t'(W_t)^{3/8}}dW_t.$$

□

Theorem 3.5. *Suppose γ is a chordal SLE$_{8/3}$ in $\mathbb{H}$ from 0 to ∞. For any $A \in \mathscr{A}_c$, we have*

$$\mathbb{P}[\gamma \cap A = \emptyset] = \Phi_A'(0)^{5/8}.$$

Proof. We may assume A has smooth boundary. Set

$$M_t = (h_t'(W_t))^{5/8}.$$

If e is a Brownian excursion with law $\mu^{\sharp}_{\mathbb{H}}(W_t,\infty)$, then $h_t'(W_t)$ is the probability of e to avoid $g_t(A)$. See Proposition 2.10. Thus, for $t < T$, $h_t'(W_t) \le 1$ and M is a bounded martingale.

$$\text{When } T = \infty, \lim_{t\to\infty} h_t'(W_t) = 1; \quad \text{when } T < \infty, \lim_{t\to T} h_t'(W_t) = 0.$$

Roughly speaking, when $T = \infty$, $g_t(A)$ will be far away from W_t as $t \to \infty$ and thus the probability for e to avoid $g_t(A)$ converges to 1; whereas, when $T < \infty$, $g_t(A)$ will be very close to W_t as $t \to \infty$. (See [LSW03] for details.)

Since M converges in L^1 and a.s. when $t \to T$, we have that

$$\mathbb{P}[\gamma \cap A = \emptyset] = \mathbb{P}[T = \infty] = \mathbb{E}[M_T] = \mathbb{E}[M_0] = \Phi_A'(0)^{5/8}.$$

□

4 Lecture 4: Chordal conformal restriction

4.1 Setup for chordal restriction sample

Let Ω be the collection of closed sets K of $\bar{\mathbb{H}}$ such that

$$K \cap \mathbb{R} = \{0\}, K \text{ is unbounded, } K \text{ is connected and } \mathbb{H} \setminus K \text{ has two connected components.}$$

And recall $\mathscr{A}_c$ in Definition 1.1. We endow Ω with the σ-field generated by the events $[K \in \Omega : K \cap A = \emptyset]$ where $A \in \mathscr{A}_c$. This family of events is closed under finite intersection, so that a probability measure on Ω is characterized by the values of $\mathbb{P}[K \cap A = \emptyset]$ for $A \in \mathscr{A}_c$: Let $\mathbb{P}, \mathbb{P}'$ are two probability measures on Ω. If $\mathbb{P}[K \cap A = \emptyset] = \mathbb{P}'[K \cap A = \emptyset]$ for all $A \in \mathscr{A}_c$, then $\mathbb{P} = \mathbb{P}'$.

Definition 4.1. *A probability measure $\mathbb{P}$ on Ω is said to satisfy chordal conformal restriction property, if the following is true:*

1. *For any $\lambda > 0$, λK has the same law as K;*
2. *For any $A \in \mathscr{A}_c$, $\Phi_A(K)$ conditioned on $[K \cap A = \emptyset]$ has the same law as K.*

Theorem 4.2.
1. *(Characterization) A chordal restriction measure is fully characterized by a positive real $\beta > 0$ such that, for every $A \in \mathscr{A}_c$,*

$$\mathbb{P}[K \cap A = \emptyset] = \Phi_A'(0)^{\beta}. \tag{4.1}$$

We denote the corresponding chordal restriction measure as $\mathbb{P}(\beta)$.

2. *(Existence) The measure $\mathbb{P}(\beta)$ exists if and only if $\beta \geq 5/8$.*

Remark 4.3. *We already know that $\mathbb{P}(\beta)$ exist for $\beta = 1$ (Proposition 2.10), $\beta = 5/8$ (Theorem 3.5), and $\beta = 5/8m + n$ for $m \geq 1, n \geq 1$.*

Homework: Suppose that K is scale-invariant and satisfies Equation (4.1) for every $A \in \mathscr{A}_c$, then check that K satisfies chordal conformal restriction property.

Proof of Theorem 4.2. Characterization. We will omit the details related to regularities and only keep the details that are related to the key idea.

Fix $x \in \mathbb{R} \setminus \{0\}$ and let $\varepsilon > 0$. We claim that the probability

$$\mathbb{P}[K \cap B(x, \varepsilon) \neq \emptyset]$$

decays like ε^2 as ε goes to zero. And the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{P}[K \cap B(x, \varepsilon) \neq \emptyset]$$

exists which we denote as $\lambda(x)$. Furthermore, $\lambda(x) \in (0, \infty)$. Since K is scale-invariant, we have that, for any $y > 0$,

$$\lambda(yx) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{P}[K \cap B(yx, \varepsilon) \neq \emptyset] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{P}[K \cap B(x, \varepsilon/y) \neq \emptyset] = y^{-2} \lambda(x).$$

Since λ is an even function, we have that, there exists $c > 0$ such that

$$\lambda(x) = cx^{-2}.$$

Since there is only one-degree of freedom, when K satisfies chordal restriction property, we must have that Equation (4.1) holds for some $\beta > 0$.

Denote $f_{x,\varepsilon} = \Phi_{\bar{\mathbb{U}}(x,\varepsilon)}$. In fact,

$$f_{x,\varepsilon}(z) = z + \frac{\varepsilon^2}{z-x} + \frac{\varepsilon^2}{x}.$$

Note that,

$$\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset] \sim \lambda(x)\varepsilon^2,$$

and that

$$1 - f'_{x,\varepsilon}(0)^\beta \sim \beta \frac{\varepsilon^2}{x^2}.$$

This implies that $\beta = c$. □

In the following of the lecture, we will first show that $\mathbb{P}(\beta)$ does not exist for $\beta < 5/8$ and then construct all $\mathbb{P}(\beta)$ for $\beta > 5/8$.

4.2 Chordal SLE$_\kappa(\rho)$ process

Definition

Suppose $\kappa > 0, \rho > -2$. Chordal SLE$_\kappa(\rho)$ process is the Loewner chain driven by W which is the solution to the following SDE:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho dt}{W_t - O_t}, \quad dO_t = \frac{2dt}{O_t - W_t}, \quad W_0 = O_0 = 0, \quad O_t \le W_t. \tag{4.2}$$

The evolution is well-defined at times when $W_t > O_t$, but a bit delicate when $W_t = O_t$. We first show the existence of the solution to this SDE.

Define Z_t as the solution to the Bessel equation

$$dZ_t = \sqrt{\kappa}dB_t + (\rho+2)\frac{dt}{Z_t}, \quad Z_0 = 0.$$

In other words, Z is $\sqrt{\kappa}$ times a Bessel process of dimension

$$d = 1 + 2(\rho+2)/\kappa.$$

And this process is well-defined for all $\rho > -2$. Note also that, for all $t \ge 0$,

$$\int_0^t \frac{du}{Z_u} < \infty.$$

Then define

$$O_t = -2\int_0^t \frac{du}{Z_u}, \quad W_t = Z_t + O_t.$$

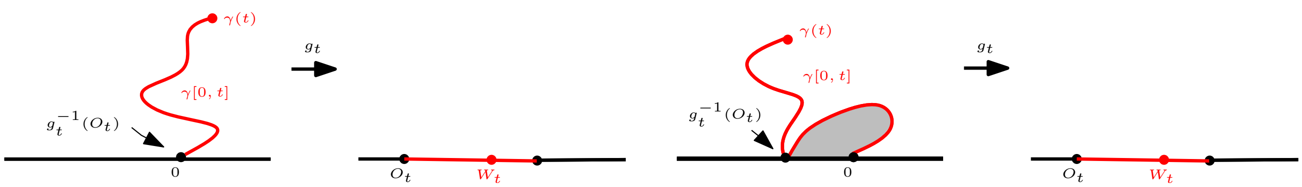
Clearly, (W_t, O_t) is a solution to Equation (4.2). And when $\rho = 0$, we get the ordinary SLE$_\kappa$.

Second, we explain the geometric meaning of the process (O_t, W_t). Recall

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

Suppose $(K_t, t \geq 0)$ is the Loewner chain generated by W, then g_t is the conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ normalized at ∞. W_t is the image of the tip, and O_t is the image of the leftmost point of $\mathbb{R} \cap K_t$. See Figure 4.1. Basic properties of $\mathrm{SLE}_\kappa(\rho)$ process: Fix $\kappa \in [0,4]$, $\rho > -2$,

- It is scale-invariant: for any $\lambda > 0$, $(\lambda^{-1} K_{\lambda^2 t}, t \geq 0)$ has the same law as K.
- $(K_t, t \geq 0)$ is generated by a continuous curve $(\gamma(t), t \geq 0)$ in $\bar{\mathbb{H}}$ from 0 to ∞.
- If $\rho \geq \kappa/2 - 2$, the dimension of the Bessel process $Z_t = W_t - O_t$ is greater than 2 and Z does not hit zero, thus almost surely $\gamma \cap \mathbb{R} = \{0\}$. If $\rho \in (-2, \kappa/2 - 2)$, almost surely $\gamma \cap \mathbb{R} \neq \{0\}$ and $K_\infty \cap \mathbb{R} = (-\infty, 0]$.[6]



(a) When $\rho \geq \kappa/2 - 2$, the curve does not hit $\mathbb{R}_-$. The hull $K_t = \gamma[0,t]$.

(b) When $\rho \in (-2, \kappa/2 - 2)$, the curve touches the boundary. The hull $K_t \neq \gamma[0,t]$.

Fig. 4.1: Geometric meaning of (O_t, W_t) in $\mathrm{SLE}_\kappa(\rho)$ process. The preimage of W_t under g_t is the tip of the curve, the preimage of O_t under g_t is the leftmost point of $K_t \cap \mathbb{R}$.

Theorem 4.4. *Fix $\rho > -2$. Let $(K_t, t \geq 0)$ be the hulls of chordal $SLE_{8/3}(\rho)$ and $K = \cup_{t \geq 0} K_t$. Then K satisfies the right-sided restriction property with exponent*

$$\beta = \frac{3\rho^2 + 16\rho + 20}{32}. \tag{4.3}$$

In other words, for every $A \in \mathscr{A}_c$ such that $A \cap \mathbb{R} \subset (0, \infty)$, we have

$$\mathbb{P}[K \cap A = \emptyset] = \Phi_A'(0)^\beta.$$

Proof. The definitions of $g_t, \tilde{g}_t, h_t$ are recalled in Figure 4.2. Set $T = \inf\{t : K_t \cap A \neq \emptyset\}$, and define, for $t < T$,

$$M_t = h_t'(W_t)^{5/8} h_t'(O_t)^{\rho(3\rho+4)/32} \left(\frac{h_t(W_t) - h_t(O_t)}{W_t - O_t} \right)^{3\rho/8}.$$

[6] When $\rho > 0$, W_t gets a push away from O_t, the curve is repelled from $\mathbb{R}_-$. When $\rho < 0$, the curve is attracted to $\mathbb{R}_-$. When $\rho < \kappa/2 - 2$, the attraction is big enough that the curve touches $\mathbb{R}_-$.

Then $(M_t, t<T)$ is a local martingale [LSW03, Lemma 8.9]:

$$\begin{aligned} dh_t(W_t) &= \left(\frac{\rho h_t'(W_t)}{W_t - O_t} - (5/3)h_t''(W_t)\right)dt + \sqrt{8/3}h_t'(W_t)dB_t, \\ dh_t'(W_t) &= \left(\frac{\rho h_t''(W_t)}{W_t - O_t} + \frac{h_t''(W_t)^2}{2h_t'(W_t)}\right)dt + \sqrt{8/3}h_t''(W_t)dB_t, \\ dh_t(O_t) &= \frac{2h_t'(W_t)^2}{h_t(O_t) - h_t(W_t)}dt, \\ dh_t'(O_t) &= \left(\frac{2h_t'(O_t)}{(O_t - W_t)^2} - \frac{2h_t'(W_t)^2 h_t'(O_t)}{(h_t(O_t) - h_t(W_t))^2}\right)dt. \end{aligned}$$

Combine these, M is a local martingale.

Since h_t' is decreasing in $(-\infty, W_t]$,

$$h_t'(W_t) \le \frac{h_t(W_t) - h_t(O_t)}{W_t - O_t} \le h_t'(O_t) \le 1.$$

In fact, there exists $\delta > 0$ such that $M_t \le h_t'(W_t)^\delta$. (We omit the proof of this point, details could be found in [LSW03, Lemma 8.10]). In particular, $M_t \le 1$ and $(M_t, t<T)$ is a bounded martingale.

$$\text{When } T=\infty,\ \lim_{t\to\infty} h_t'(W_t) = 1,\ \lim_{t\to\infty} M_t = 1; \quad \text{when } T<\infty,\ \lim_{t\to T} h_t'(W_t) = 0,\ \lim_{t\to T} M_t = 0.$$

Thus

$$\mathbb{P}[K \cap A = \emptyset] = \mathbb{P}[T=\infty] = \mathbb{E}[M_T] = M_0 = \Phi_A'(0)^\beta$$

where β is given by Equation (4.3). □

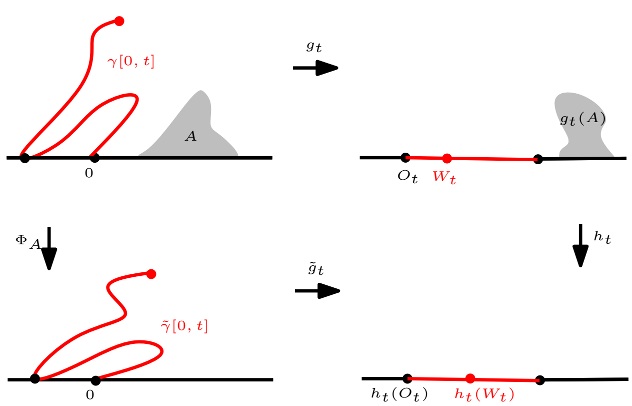


Fig. 4.2: Φ_A is the conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ with $\Phi_A(0)=0$, $\Phi_A(\infty)=\infty$, and $\Phi_A(z)/z \to 1$ as $z\to\infty$. g_t is the conformal map from $\mathbb{H}\setminus\gamma[0,t]$ onto $\mathbb{H}$ normalized at infinity. $\tilde g_t$ is the conformal map from $\mathbb{H}\setminus\tilde\gamma[0,t]$ onto $\mathbb{H}$ normalized at infinity. h_t is the conformal map from $\mathbb{H}\setminus g_t(A)$ onto $\mathbb{H}$ such that $h_t\circ g_t = \tilde g_t \circ \Phi_A$.

Setup for right-sided restriction property

Let Ω^+ be the collection of closed sets K of $\bar{\mathbb{H}}$ such that

$$K \cap \mathbb{R} = (-\infty, 0], K \text{ is connected and } \mathbb{H} \setminus K \text{ is connected.}$$

And recall $\mathscr{A}_c$ in Definition 1.1. And let $\mathscr{A}_c^+$ denote the set of $A \in \mathscr{A}_c$ such that $A \cap \mathbb{R} \subset (0, \infty)$. We endow Ω^+ with the σ-field generated by the events $[K \in \Omega^+ : K \cap A = \emptyset]$ where $A \in \mathscr{A}_c^+$.

Definition 4.5. *A probability measure $\mathbb{P}$ on Ω^+ is said to satisfy right-sided restriction property, if the following is true*

1. *For any $\lambda > 0$, λK has the same law as K;*
2. *For any $A \in \mathscr{A}_c^+$, $\Phi_A(K)$ conditioned on $[K \cap A = \emptyset]$ has the same law as K.*

Similar to the proof of Theorem 4.2, we know that, if $\mathbb{P}$ satisfies the right-sided restriction property, then there exists $\beta > 0$ such that

$$\mathbb{P}[K \cap A = \emptyset] = \Phi_A'(0)^\beta, \quad \text{for all } A \in \mathscr{A}_c^+.$$

Remark 4.6. *Theorem 4.4 states that $SLE_{8/3}(\rho)$ has the same law as the right boundary of the right-sided restriction sample with exponent β which is related to ρ through Equation (4.3). Note that when ρ spans $(-2, \infty)$, β spans $(0, \infty)$. In particular, Theorem 4.4 also states the existence of right-sided restriction measure for all $\beta > 0$.*

Remark 4.7. *If $\beta \geq 5/8$, the right boundary of (two-sided) restriction measure $\mathbb{P}(\beta)$ has the same law as $SLE_{8/3}(\rho)$ where*

$$\rho = \rho(\beta) = \frac{1}{3}(-8 + 2\sqrt{24\beta + 1}). \tag{4.4}$$

In particular, the right boundary of a Brownian excursion has the law of $SLE_{8/3}(2/3)$, the right boundary of the union of two independent Brownian excursions has the law of $SLE_{8/3}(2)$.

Remark 4.8. *Recall Theorem 2.15, suppose $(e_j, j \in J)$ is a Poisson point process with intensity $\pi\beta\mu^{exc}_{\mathbb{H},\mathbb{R}_-}$, and set $\Sigma = \cup_j e_j$, then the right boundary of Σ has the same law as chordal $SLE_{8/3}(\rho)$ where $\rho = \rho(\beta)$ given by Equation (4.4) for all $\beta > 0$.*

Now we can prove one part of Theorem 4.2.

Proof of Theorem 4.2, $\mathbb{P}(\beta)$ does not exist for $\beta < 5/8$. Suppose the (two-sided) chordal restriction measure $\mathbb{P}(\beta)$ exists for some $\beta < 5/8$. The right boundary γ of K is $\text{SLE}_{8/3}(\rho)$ for $\rho = \rho(\beta) < 0$.

On the one hand, K is symmetric with respect to the imaginary axis, thus the probability of i staying to the right of γ is less than $1/2$.

On the other hand, since $\rho < 0$, the probability of i staying to the right of γ is strictly larger than the probability of i staying to the right of $\text{SLE}_{8/3}$ which equals $1/2$.

Contradiction. □

4.3 Construction of $\mathbb{P}(\beta)$ for $\beta > 5/8$

In the previous definition of $\mathrm{SLE}_\kappa(\rho)$ process, there is a repulsion (when $\rho > 0$) or attraction (when $\rho < 0$) from $\mathbb{R}_-$. We will denote this process as $\mathrm{SLE}^L_\kappa(\rho)$. And symmetrically, we denote $\mathrm{SLE}^R_\kappa(\rho)$ as the same process only except that the repulsion or attraction is from $\mathbb{R}_+$. Precisely, $\mathrm{SLE}^R_\kappa(\rho)$ is the Loewner chain driven by W which is the solution to the following SDE:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho dt}{W_t - O_t}, \quad dO_t = \frac{2dt}{O_t - W_t}, \quad W_0 = O_0 = 0, \quad O_t \geq W_t. \tag{4.5}$$

$\mathrm{SLE}^R_\kappa(\rho)$ can also be viewed as the image of $\mathrm{SLE}^L_\kappa(\rho)$ under the reflection with respective to the imaginary axis.

From Theorem 4.4, we know that $\mathrm{SLE}^L_{8/3}(\rho)$ satisfies right-sided restriction property and $\mathrm{SLE}^R_{8/3}(\rho)$ satisfies left-sided restriction property. The idea to construct K whose law is $\mathbb{P}(\beta)$ for $\beta > 5/8$ is the following: we first run an $\mathrm{SLE}^L_{8/3}(\rho)$ as the right-boundary of K, and then given the right boundary, we run the left boundary according to the conditional law.

Proposition 4.9. *Fix $\beta > 5/8$, and $\rho = \rho(\beta) > 0$ where $\rho(\beta)$ is given by Equation (4.4). Suppose γ^R is a chordal $SLE^L_{8/3}(\rho)$ process in $\bar{\mathbb{H}}$ from 0 to ∞. Given γ^R, in the left-connected component of $\mathbb{H}\setminus\gamma^R$, sample an $SLE^R_{8/3}(\rho-2)$ from 0 to ∞ which is denoted as γ^L. Let K be the closure of the union of the domains between γ^L and γ^R. Then K has the law of $\mathbb{P}(\beta)$.*

Proof. We only need to check, for all $A \in \mathscr{A}_c$,

$$\mathbb{P}[K\cap A = \emptyset] = \Phi'_A(0)^\beta.$$

From the construction, we know that this is true for $A \in \mathscr{A}_c^+$. We only need to prove it for $A \in \mathscr{A}_c$ such that $A\cap\mathbb{R} \subset (-\infty, 0)$. Let $(g_t, t \geq 0)$ be the solution of the Loewner chain for the process γ^R and $(O_t, W_t, t \geq 0)$ be the solution of the SDE (4.2). Set $T = \inf\{t : \gamma^R(t) \in A\}$. And,for $t < T$, let h_t be the conformal map from $\mathbb{H}\setminus g_t(A)$ onto $\mathbb{H}$ normalized at ∞. See Figure 4.3. Recall that

$$M_t = h'_t(W_t)^{5/8} h'_t(O_t)^{\rho(3\rho+4)/32}\left(\frac{h_t(W_t)-h_t(O_t)}{W_t - O_t}\right)^{3\rho/8}$$

is a local martingale. And that

$$0 \leq h'_t(O_t) \leq \frac{h_t(W_t)-h_t(O_t)}{W_t - O_t} \leq h'_t(W_t) \leq 1.$$

Since $\rho > 0$, we have that $M_t \leq h'_t(W_t)^\beta$ and thus M is a bounded martingale.

If $T < \infty$, then

$$h'_t(W_t) \to 0, \quad \text{and } M_t \to 0 \text{ as } t \to T.$$

If $T = \infty$, then

$$h'_t(W_t) \to 1, \quad \frac{h_t(W_t)-h_t(O_t)}{W_t - O_t} \to 1,$$

and (apply Theorem 4.4 to $\mathrm{SLE}^R_{8/3}(\rho-2)$)

$$h'_t(O_t)^{\rho(3\rho+4)/32} \to \mathbb{P}[\gamma^L\cap A = \emptyset\,|\,\gamma^R] \text{ as } t\to\infty.$$

Thus,

$$\mathbb{P}[K\cap A = \emptyset] = E[1_{T=\infty}\mathbb{E}[1_{K\cap A=\emptyset}\,|\,\gamma^R]] = E[M_T] = M_0.$$

□

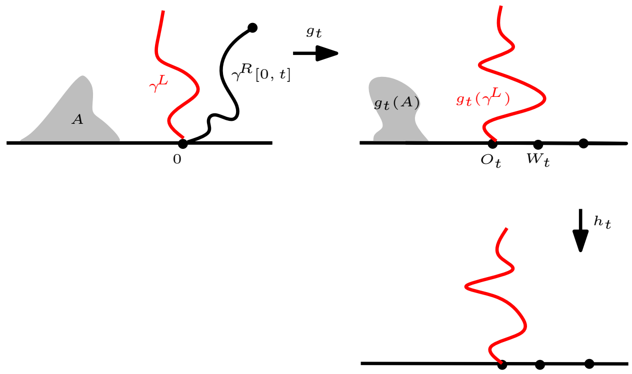


Fig. 4.3: g_t is the conformal map from $\mathbb{H}\setminus\gamma^R[0,t]$ onto $\mathbb{H}$ normalized at ∞. h_t is the conformal map from $\mathbb{H}\setminus g_t(A)$ onto $\mathbb{H}$ normalized at ∞.

4.4 Half-plane intersection exponents $\tilde{\xi}(\beta_1,\ldots,\beta_p)$

Recall that

$$\tilde{\xi}(\beta_1,\ldots,\beta_p)=\frac{1}{24}\left((\sqrt{24\beta_1+1}+\cdots+\sqrt{24\beta_p+1}-(p-1))^2-1\right),$$

and define

$$\hat{\xi}(\beta_1,\ldots,\beta_p)=\tilde{\xi}(\beta_1,\ldots,\beta_p)-\beta_1-\cdots-\beta_p. \tag{4.6}$$

For $x\in\mathbb{C}$ and a subset $K\subset\mathbb{C}$, denote

$$x+K=\{x+z:z\in K\}.$$

Proposition 4.10. *Suppose $K_1,\ldots,K_p$ are p independent chordal restriction samples with exponents $\beta_1,\ldots,\beta_p\geq 5/8$ respectively. Fix $R>0$. Let $\varepsilon>0$ be small. Set $x_j=j\varepsilon$ for $j=1,\ldots,p$. Then*

$$\mathbb{P}[(x_{j_1}+K_{j_1}\cap\mathbb{U}(0,R))\cap(x_{j_2}+K_{j_2}\cap\mathbb{U}(0,R))=\emptyset,1\leq j_1<j_2\leq p]\approx\varepsilon^{\hat{\xi}(\beta_1,\ldots,\beta_p)},\quad \textit{as } \varepsilon\to 0.$$

In the following theorem, we will consider the law of $K_1,\ldots,K_p$ conditioned on "non-intersection". Since the event of "non-intersection" has zero probability, we need to explain the precise meaning: the conditioned law would be obtained through a limiting procedure: first consider the law of $K_1,\ldots,K_p$ conditioned on

$$[(x_{j_1}+K_{j_1}\cap\mathbb{U}(0,R))\cap(x_{j_2}+K_{j_2}\cap\mathbb{U}(0,R))=\emptyset,1\leq j_1<j_2\leq p],$$

and then let $R\to\infty$ and $\varepsilon\to 0$.

Theorem 4.11. *Fix $\beta_1,\ldots,\beta_p\geq 5/8$. Suppose $K_1,\ldots,K_p$ are p independent chordal restriction samples with exponents $\beta_1,\ldots,\beta_p$ respectively. Then the "fill-in" of the union of these p sets conditioned on "non-intersection" has the same law as chordal restriction sample of exponent $\tilde{\xi}(\beta_1,\ldots,\beta_p)$.*

For Proposition 4.10 and Theorem 4.11, we only need to show the results for $p=2$ and other p can be proved by induction. Proposition 4.10 for $p=2$ is a direct consequence of the following lemma.

Lemma 4.12. *Suppose K is a right-sided restriction sample with exponent $\beta > 0$. Let γ be an independent chordal $SLE^R_{8/3}(\rho)$ process for some $\rho > -2$. Fix $t > 0$ and let $\varepsilon > 0$ be small, we have*

$$\mathbb{P}[\gamma[0,t] \cap (K-\varepsilon) = \emptyset] \approx \varepsilon^{\frac{3}{16}\bar{\rho}(\rho+2)} \quad as\ \varepsilon \to 0$$

where

$$\bar{\rho} = \frac{2}{3}(\sqrt{24\beta+1} - 1).$$

Note that, if $\beta_1 = \beta$, $\beta_2 = (3\rho^2 + 16\rho + 20)/32$, we have

$$\frac{3}{16}\bar{\rho}(\rho+2) = \hat{\tilde{\xi}}(\beta_1, \beta_2).$$

Proof. Let $(g_t, t \ge 0)$ be the Loewner chain for γ and (O_t, W_t) be the solution to the SDE. Precisely,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z;$$

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho dt}{W_t - O_t}, \quad dO_t = \frac{2dt}{O_t - W_t}, \quad W_0 = O_0 = 0, \quad O_t \ge W_t.$$

Given $\gamma[0,t]$, since K satisfies right-sided restriction property, we have that

$$\mathbb{P}[\gamma[0,t] \cap (K-\varepsilon) = \emptyset \,|\, \gamma[0,t]] = g_t'(-\varepsilon)^{\beta}.$$

Define

$$M_t = g_t'(-\varepsilon)^{\bar{\rho}(3\bar{\rho}+4)/32} (W_t - g_t(-\varepsilon))^{3\bar{\rho}/8} (O_t - g_t(-\varepsilon))^{3\bar{\rho}\rho/16}.$$

One can check that M is a local martingale. Thus

$$\mathbb{P}[(K-\varepsilon) \cap \gamma[0,t] = \emptyset] = \mathbb{E}[g_t'(-\varepsilon)^{\beta}] \approx \mathbb{E}[M_t] = M_0.$$

□

Proof of Theorem 4.11. Assume $p = 2$. For any $A \in \mathscr{A}_c$, we need to estimate the following probability for $\varepsilon > 0$ small:

$$\mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset \,|\, (K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset].$$

Since K_i satisfies chordal conformal restriction property, conditioned on $[K_i \cap A = \emptyset]$, the conditional law of $\Phi_A(K_i)$ has the same law as K_i, for $i = 1, 2$. Thus

$$\begin{aligned}
&\lim_{R\to\infty,\varepsilon\to 0} \mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset \,|\, (K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset] \\
&= \lim_{R\to\infty,\varepsilon\to 0} \frac{\mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset, (K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset]}{\mathbb{P}[(K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset]} \\
&= \lim_{R\to\infty,\varepsilon\to 0} \Phi_A'(0)^{\beta_1} \Phi_A'(0)^{\beta_2} \frac{\mathbb{P}[(K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset \,|\, K_1 \cap A = \emptyset, K_2 \cap A = \emptyset]}{\mathbb{P}[(K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset]} \\
&= \lim_{R\to\infty,\varepsilon\to 0} \Phi_A'(0)^{\beta_1} \Phi_A'(0)^{\beta_2} \frac{\mathbb{P}[(K_1 \cap \mathbb{U}(0,R) + \Phi_A(\varepsilon)) \cap (K_2 \cap \mathbb{U}(0,R) + \Phi_A(2\varepsilon)) = \emptyset]}{\mathbb{P}[(K_1 \cap \mathbb{U}(0,R) + \varepsilon) \cap (K_2 \cap \mathbb{U}(0,R) + 2\varepsilon) = \emptyset]} \\
&= \lim_{\varepsilon\to 0} \Phi_A'(0)^{\beta_1} \Phi_A'(0)^{\beta_2} \frac{\Phi_A(\varepsilon)^{\hat{\tilde{\xi}}(\beta_1,\beta_2)}}{\varepsilon^{\hat{\tilde{\xi}}(\beta_1,\beta_2)}} \\
&= \Phi_A'(0)^{\beta_1+\beta_2+\hat{\tilde{\xi}}(\beta_1,\beta_2)}.
\end{aligned}$$

□

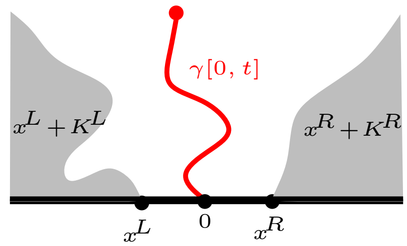


Fig. 4.4: K^L is a right-sided restriction sample with exponent $\beta^L > 0$ and K^R is a left-sided restriction sample with exponent $\beta^R > 0$. γ is an $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ with force points $(x^L;x^R)$ with $x^L < 0 < x^R$.

4.5 Related calculation

Chordal $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process

Suppose $\kappa > 0, \rho^L > -2, \rho^R > -2$ and $x^L < 0 < x^R$. Chordal $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process with force points $(x^L;x^R)$ is the Loewner chain driven by W which is the solution to the following SDE:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho^L dt}{W_t - O_t^L} + \frac{\rho^R dt}{W_t - O_t^R},$$

$$dO_t^L = \frac{2dt}{O_t^L - W_t}, \quad dO_t^R = \frac{2dt}{O_t^R - W_t}, \quad W_0 = 0, O_0^L = x^L, O_0^R = x^R.$$

There exists piecewise unique solution to the above SDE. And there exists almost surely a continuous curve γ in $\bar{\mathbb{H}}$ from 0 to ∞ associated to the $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process with force points $(x^L;x^R)$. Note that, for small time t when x^L, x^R are not swallowed by K_t, x^L (resp. x^R) is the preimage of O_t^L (resp. O_t^R) under g_t.

It is worthwhile to point out the relation between $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ processes with different ρ's. Fix $\kappa > 0$, $x^L < 0 < x^R$. Let $\rho^L > -2$, $\rho^R > -2$, $\tilde\rho^L > -2$, $\tilde\rho^R > -2$. Define

$$\begin{aligned} M_t = {} & g_t'(x^L)^{(\tilde\rho^L-\rho^L)(\tilde\rho^L+\rho^L+4-\kappa)/(4\kappa)} g_t'(x^R)^{(\tilde\rho^R-\rho^R)(\tilde\rho^R+\rho^R+4-\kappa)/(4\kappa)} \\ & \times |g_t(x^L) - W_t|^{(\tilde\rho^L-\rho^L)/\kappa} |g_t(x^R) - W_t|^{(\tilde\rho^R-\rho^R)/\kappa} \\ & \times |g_t(x^L) - g_t(x^R)|^{(\tilde\rho^L\tilde\rho^R-\rho^L\rho^R)/(2\kappa)}. \end{aligned}$$

Then M is a local martingale under the measure of $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process (see [SW05, Theorem 6]). Moreover, the measure weighted by M/M_0 (with an appropriate stopping time) is the same as the law of $\mathrm{SLE}_\kappa(\tilde\rho^L;\tilde\rho^R)$ process.

General calculation related to Lemma 4.12

In fact, the estimate in Lemma 4.12 is a special case of the following estimate. Suppose K^L is a right-sided restriction sample with exponent $\beta^L > 0$, and K^R is a left-sided restriction sample with exponent $\beta^R > 0$. Let γ be a chordal $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ in $\mathbb{H}$ from 0 to ∞ with force points $(x^L;x^R)$ where $\kappa > 0, \rho^L > -2, \rho^R > -2$, and $x^L < 0 < x^R$. See Figure 4.4. Suppose K^L, K^R, γ are independent and let $\bar\rho^L > \rho^L, \bar\rho^R > \rho^R$ be the solutions to the equations

$$\beta^L = \frac{1}{4\kappa}(\bar\rho^L - \rho^L)(\bar\rho^L + \rho^L + 4 - \kappa), \quad \beta^R = \frac{1}{4\kappa}(\bar\rho^R - \rho^R)(\bar\rho^R + \rho^R + 4 - \kappa).$$

Then, for fixed time $t > 0$,

$$\begin{aligned}&\mathbb{P}[\gamma[0,t]\cap(x^L+K^L)=\emptyset,\gamma[0,t]\cap(x^R+K^R)=\emptyset]\\ \approx\ & |x^L|^{(\bar\rho^L-\rho^L)/\kappa}|x^R|^{(\bar\rho^R-\rho^R)/\kappa}|x^R-x^L|^{(\bar\rho^L\bar\rho^R-\rho^L\rho^R)/(2\kappa)}\quad \text{as } x^L,x^R\to 0.\end{aligned}$$

Proof. Let $(g_t, t \geq 0)$ be the Loewner chain for γ. Given $\gamma[0,t]$, we have that

$$\mathbb{P}[(x^L+K^L)\cap\gamma[0,t]=\emptyset\,|\,\gamma[0,t]]=g_t'(x^L)^{\beta^L},\quad \mathbb{P}[(x^R+K^R)\cap\gamma[0,t]=\emptyset\,|\,\gamma[0,t]]=g_t'(x^R)^{\beta^R}.$$

Note that

$$M_t=g_t'(x^L)^{\beta^L}g_t'(x^R)^{\beta^R}|g_t(x^L)-W_t|^{(\bar\rho^L-\rho^L)/\kappa}|g_t(x^R)-W_t|^{(\bar\rho^R-\rho^R)/\kappa}|g_t(x^R)-g_t(x^L)|^{(\bar\rho^L\bar\rho^R-\rho^L\rho^R)/(2\kappa)}$$

is a local martingale. Thus,

$$\begin{aligned}&\mathbb{P}[\gamma[0,t]\cap(x^L+K^L)=\emptyset,\gamma[0,t]\cap(x^R+K^R)=\emptyset]\\ =\ & \mathbb{E}[g_t'(x^L)^{\beta^L}g_t'(x^R)^{\beta^R}]\\ \approx\ & \mathbb{E}[M_t]=M_0.\end{aligned}$$

□

5 Lecture 5: Radial SLE

5.1 Radial Loewner chain

Capacity

Consider a compact subset K of $\bar{\mathbb{U}}$ such that $0 \in \mathbb{U} \setminus K$ and $\mathbb{U} \setminus K$ is simply connected. Then there exists a unique conformal map g_K from $\mathbb{U} \setminus K$ onto $\mathbb{U}$ normalized at the origin, i.e. $g_K(0) = 0, g_K'(0) > 0$. We call $a(K) := \log g_K'(0)$ as the capacity of K in $\mathbb{U}$ seen from the origin.

Lemma 5.1. *a is non-negative increasing function.*

Proof. a is non-negative: Denote $U = \mathbb{U} \setminus K$. $\log g_K(z)/z$ is an analytic function on $\mathbb{U} \setminus \{0\}$ and the origin is removable: we can define the function equals $\log g_K'(0)$ at the origin. Then $h(z) = \log |g_K(z)/z|$ is a harmonic function on U. Thus it attains its min on ∂U. For $z \in \partial D$, $h(z) \geq 0$. Therefore $h(z) \geq 0$ for all $z \in U$. In particular, $h(0) \geq 0$.

a is increasing: Suppose $K \subset K'$. Define $g_1 = g_K$ and let g_2 be the conformal map from $\mathbb{U} \setminus g_K(K' \setminus K)$ onto $\mathbb{U}$ normalized at the origin. Then $g_{K'} = g_2 \circ g_1$. Thus

$$a(K') = \log g_2'(0) + \log g_1'(0) \geq \log g_1'(0) = a(K).$$

□

Remark 5.2. *If we denote $d(0,K)$ as Euclidean distance from the origin to K, by Koebe $1/4$-Theorem, we have that*

$$\frac{1}{4} e^{-a(K)} \leq d(0,K) \leq e^{-a(K)}.$$

Loewner chain

Suppose $(W_t, t \geq 0)$ is a continuous real function with $W_0 = 0$. Define for $z \in \bar{\mathbb{U}}$, the function $g_t(z)$ as the solution to the ODE

$$\partial_t g_t(z) = g_t(z) \frac{e^{iW_t} + g_t(z)}{e^{iW_t} - g_t(z)}, \quad g_0(z) = z.$$

The solution is well-defined as long as $e^{iW_t} - g_t(z)$ does not hit zero. Define Define

$$T(z) = \sup\{t > 0 : \min_{s \in [0,t]} |e^{iW_s} - g_s(z)| > 0\}.$$

This is the largest time up to which $g_t(z)$ is well-defined. Set

$$K_t = \{z \in \bar{\mathbb{U}} : T(z) \leq t\}, \quad U_t = \mathbb{U} \setminus K_t.$$

We can check that

- g_t is a conformal map from U_t onto $\mathbb{U}$ normalized at the origin.
- For each t, $g_t'(0) = e^t$. In other words, $a(K_t) = t$.

The family $(K_t, t \geq 0)$ is called the **radial Loewner chain** driven by $(W_t, t \geq 0)$.

5.2 Radial SLE

Definition

Radial SLE_κ for $\kappa \geq 0$ is the radial Loewner chain driven by $W_t = \sqrt{\kappa} B_t$ where B is a 1-dimensional BM starting from $B_0 = 0$.

Lemma 5.3. *Radial SLE satisfies domain Markov property: For any stopping time T, the process $(g_T(K_{t+T} \setminus K_T)e^{-iW_T}, t \geq 0)$ is independent of $(K_s, 0 \leq s \leq T)$ and has the same law as K.*

Proposition 5.4. *For $\kappa \in [0,4]$, radial SLE_κ is almost surely generated by a simple continuous curve, i.e. there exists a simple continuous curve γ such that, for all $t \geq 0$, $K_t = \gamma[0,t]$. See Figure 5.1. And almost surely,*

$$\lim_{t\to\infty} \gamma(t) = 0.$$

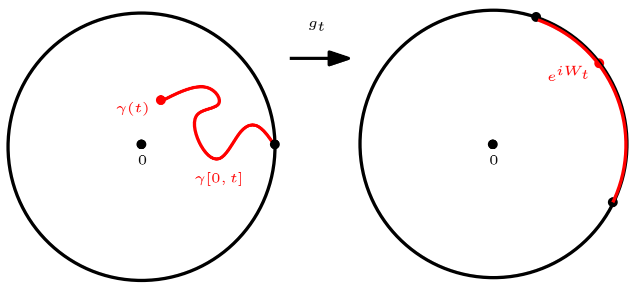


Fig. 5.1: g_t is the conformal map from $\mathbb{U} \setminus \gamma[0,t]$ onto $\mathbb{U}$ normalized at the origin. And the tip of the curve $\gamma(t)$ is the preimage of e^{iW_t} under g_t: $\gamma(t) = g_t^{-1}(e^{iW_t})$.

Restriction property of radial $\mathrm{SLE}_{8/3}$

Recall Definition 1.2. Suppose $A \in \mathscr{A}_r$, and Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ such that $\Phi_A(0) = 0, \Phi_A(1) = 1$. Let γ be a radial $\mathrm{SLE}_{8/3}$, we will compute the probability

$$\mathbb{P}[\gamma \cap A = \emptyset].$$

Similar as the chordal case, define

$$T = \inf\{t : \gamma(t) \in A\}. \quad \text{For } t < T, \tilde{\gamma}[0,t] := \Phi_A(\gamma[0,t]).$$

Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ with $\Phi_A(0) = 0, \Phi_A(1) = 1$. g_t is the conformal map from $\mathbb{U} \setminus \gamma[0,t]$ onto $\mathbb{U}$ normalized at the origin. Define $\tilde{g}_t$ as the conformal map from $\mathbb{U} \setminus \tilde{\gamma}[0,t]$ onto $\mathbb{U}$ normalized at the origin and h_t as the conformal map from $\mathbb{U} \setminus g_t(A)$ onto $\mathbb{U}$ such that Equation (5.1) holds. See Figure 5.2.

$$h_t \circ g_t = \tilde{g}_t \circ \Phi_A. \tag{5.1}$$

Proposition 5.5. *When $\kappa = 8/3$, the process*

$$M_t = |h_t'(0)|^{5/48} |h_t'(e^{iW_t})|^{5/8}, \quad t < T$$

is a local martingale.

Proof. Define
$$\phi_t(z) = -i\log h_t(e^{iz})$$
where log denotes the branch of the logarithm such that $-i\log h_t(e^{iW_t}) = W_t$. Then
$$h_t(e^{iz}) = e^{i\phi_t(z)}, \quad h_t'(e^{iW_t}) = \phi_t'(W_t).$$
Define
$$a(t) = a(K_t \cap A) = a(A) + a(\tilde{K}_t).$$
A simple time change shows that
$$\partial_t \tilde{g}_t(z) = \partial_t a\, \tilde{g}_t(z) \frac{e^{i\tilde{W}_t} + \tilde{g}_t(z)}{e^{i\tilde{W}_t} - \tilde{g}_t(z)}.$$
Plugin $h_t \circ g_t = \tilde{g}_t \circ \Phi_A$, we have
$$\partial_t h_t(z) + h_t'(z) z \frac{e^{iW_t} + z}{e^{iW_t} - z} = \partial_t a\, h_t(z) \frac{e^{iW_t} + h_t(z)}{e^{iW_t} - h_t(z)}. \tag{5.2}$$
We can first decide $\partial_t a$: multiply $e^{iW_t} - h_t(z)$ to both sides of Equation (5.2) and then let $z \to e^{iW_t}$. We have
$$\partial_t a = h_t'(e^{iW_t})^2 = \phi_t'(W_t)^2.$$
Denote
$$X_1 = \phi_t'(W_t), \quad X_2 = \phi_t''(W_t), \quad X_3 = \phi_t'''(W_t).$$
Then Equation (5.2) becomes
$$\partial_t h_t(z) = X_1^2 h_t(z) \frac{e^{iW_t} + h_t(z)}{e^{iW_t} - h_t(z)} - h_t'(z) z \frac{e^{iW_t} + z}{e^{iW_t} - z}.$$
Plugin the relation $h_t(e^{iz}) = e^{i\phi_t(z)}$, we have that
$$\partial_t \phi_t(z) = X_1^2 \cot(\frac{\phi_t(z) - W_t}{2}) - \phi_t'(z) \cot(\frac{z - W_t}{2}). \tag{5.3}$$
Differentiate Equation (5.3) with respect to z, we have
$$\partial_t \phi_t'(z) = -\frac{1}{2} X_1^2 \phi_t'(z) \csc^2(\frac{\phi_t(z) - W_t}{2}) - \phi_t''(z) \cot(\frac{z - W_t}{2}) + \frac{1}{2} \phi_t'(z) \csc^2(\frac{z - W_t}{2}).$$
Let $z \to W_t$,
$$\partial_t \phi_t'(W_t) = \frac{X_2^2}{2X_1} - \frac{4}{3} X_3 + \frac{X_1 - X_1^3}{6}.$$
Thus
$$dh_t'(e^{iW_t}) = d\phi_t'(W_t) = \sqrt{\frac{8}{3}} X_2 dB_t + (\frac{X_2^2}{2X_1} + \frac{X_1 - X_1^3}{6}) dt. \tag{5.4}$$
For the term $h_t'(0)$, we have that
$$h_t'(0) = \exp(a(g_t(A))) = \exp(a(t) - t) = \exp(a(A) + \int_0^t \phi_s'(W_s)^2 ds - t),$$

thus

$$dh_t'(0) = h_t'(0)(X_1^2 - 1)dt. \tag{5.5}$$

Combine Equations (5.4) and (5.5), we have that

$$dM_t = \frac{5}{8} M_t \frac{X_2}{X_1} dW_t.$$

□

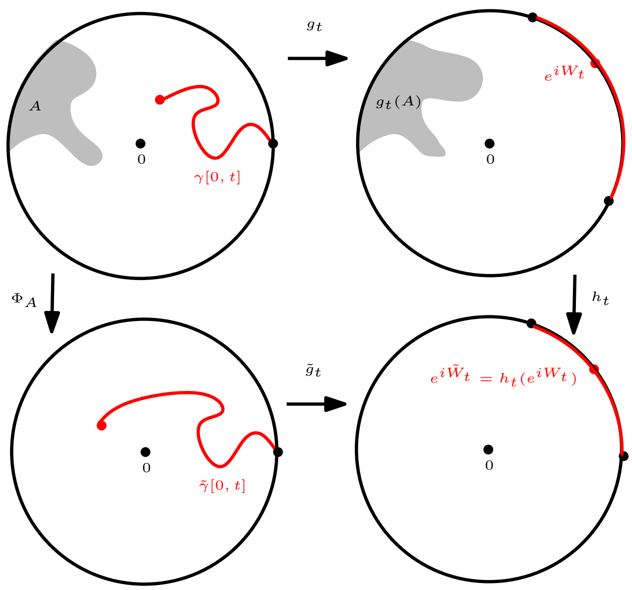


Fig. 5.2: Φ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ with $\Phi_A(0) = 0, \Phi_A(1) = 1$. g_t is the conformal map from $\mathbb{U}\setminus\gamma[0,t]$ onto $\mathbb{U}$ normalized at the origin. Define $\tilde{g}_t$ as the conformal map from $\mathbb{U}\setminus\tilde{\gamma}[0,t]$ onto $\mathbb{U}$ normalized at the origin and h_t as the conformal map from $\mathbb{U}\setminus g_t(A)$ onto $\mathbb{U}$ such that $h_t \circ g_t = \tilde{g}_t \circ \Phi_A$.

Theorem 5.6. *Suppose γ is a radial $SLE_{8/3}$ in $\mathbb{U}$ from 1 to 0. Then for any $A \in \mathscr{A}_r$, we have*

$$\mathbb{P}[\gamma \cap A = \emptyset] = |\Phi_A'(0)|^{5/48} \Phi_A'(1)^{5/8}.$$

Proof. Suppose M is the local martingale defined in Proposition 5.5. Note that

$$M_0 = |\Phi_A'(0)|^{5/48} \Phi_A'(1)^{5/8}.$$

Define $T = \inf\{t : K_t \cap A \neq \emptyset\}$. In fact, $|\Phi_A'(0)|\Phi_A'(1)^2 \leq 1$ for any $A \in \mathscr{A}_r$, thus M is a bounded martingale.

If $T < \infty$, $\lim_{t\to T} h_t'(e^{iW_t}) = 0$, and $\lim_{t\to T} M_t = 0$.

If $T = \infty$, $\lim_{t\to\infty} h_t'(e^{iW_t}) = 1$, $\lim_{t\to\infty} h_t'(0) = 1$, and $\lim_{t\to\infty} M_t = 1$.

Thus,

$$\mathbb{P}[\gamma \cap A = \emptyset] = \mathbb{P}[T = \infty] = E[M_T] = M_0.$$

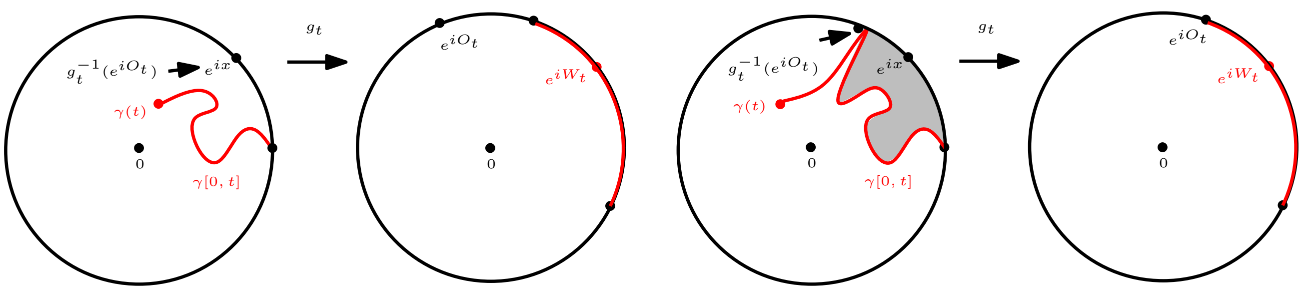
□

5.3 Radial SLE$_\kappa(\rho)$ process

Fix $\kappa > 0$, $\rho > -2$. Radial SLE$_\kappa(\rho)$ process is the radial Loewner chain driven by W which is the solution to the following SDE:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho}{2}\cot(\frac{W_t - O_t}{2})dt, \quad dO_t = -\cot(\frac{W_t - O_t}{2})dt, \quad W_0 = 0, O_0 = x \in (0, 2\pi). \tag{5.6}$$

When $\kappa > 0, \rho > -2$, there exists a piecewise unique solution to the SDE (5.6). There exists almost surely a continuous curve γ in $\bar{\mathbb{U}}$ from 1 to 0 so that $(K_t, t \geq 0)$ is generated by γ. When $\kappa \in [0,4]$ and $\rho \geq \kappa/2 - 2$, γ is a simple curve and $K_t = \gamma[0,t]$. When $\kappa \in [0,4], \rho \in (-2, \kappa/2-2)$, γ almost surely hits the boundary. The tip $\gamma(t)$ is the preimage of e^{iW_t} under g_t. And e^{ix} (when it is not swallowed by K_t) is the preimage of e^{iO_t} under g_t. When e^{ix} is swallowed by K_t, then the preimage of e^{iO_t} under g_t is the last point (before time t) on the curve that is on the boundary. See Figure 5.3.



(a) When $\rho \geq \kappa/2 - 2$, the curve does not hit the boundary. The hull $K_t = \gamma[0,t]$.

(b) When $\rho \in (-2, \kappa/2-2)$, the curve touches the boundary. The hull $K_t \neq \gamma[0,t]$.

Fig. 5.3: Geometric meaning of (O_t, W_t) in radial SLE$_\kappa(\rho)$ process. The preimage of e^{iW_t} under g_t is the tip of the curve, the preimage of e^{iO_t} under g_t is the last point on the curve that is on the boundary.

Let $x \to 0+$ (resp. $x \to 2\pi-$), the process has a limit, and we call this limit as radial SLE$^R_\kappa(\rho)$ (resp. SLE$^L_\kappa(\rho)$) in $\bar{\mathbb{U}}$ from 1 to 0.

Suppose γ is an SLE$^L_{8/3}(\rho)$ process for some $\rho > -2$. For any $A \in \mathscr{A}_r$, we want to analyze the image of γ under Φ_A. Define $T = \inf\{t : \gamma(t) \in A\}$. For $t < T$, Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ with $\Phi_A(0) = 0, \Phi_A(1) = 1$, g_t is the conformal map from $\mathbb{U} \setminus \gamma[0,t]$ onto $\mathbb{U}$ normalized at the origin. Define $\tilde{g}_t$ as the conformal map from $\mathbb{U} \setminus \tilde{\gamma}[0,t]$ onto $\mathbb{U}$ normalized at the origin and h_t as the conformal map from $\mathbb{U} \setminus g_t(A)$ onto $\mathbb{U}$ such that $h_t \circ g_t = \tilde{g}_t \circ \Phi_A$. See Figure 5.4. Denote

$$\theta_t = \frac{W_t - O_t}{2}, \quad \vartheta_t = \frac{1}{2}\arg(h_t(e^{iW_t})/h_t(e^{iO_t})).$$

Proposition 5.7. *Define*

$$M_t = |h_t'(0)|^\alpha \times |h_t'(e^{iW_t})|^{5/8} \times |h_t'(e^{iO_t})|^{\rho(3\rho+4)/32} \times (\frac{\sin\vartheta_t}{\sin\theta_t})^{3\rho/8}$$

where

$$\alpha = \frac{5}{48} + \frac{3}{64}\rho(\rho+4).$$

Then M is a local martingale. Note that, if we set

$$\beta = \frac{5}{8} + \frac{1}{32}\rho(3\rho+4) + \frac{3}{8}\rho,$$

we have $\alpha = \xi(\beta)$.

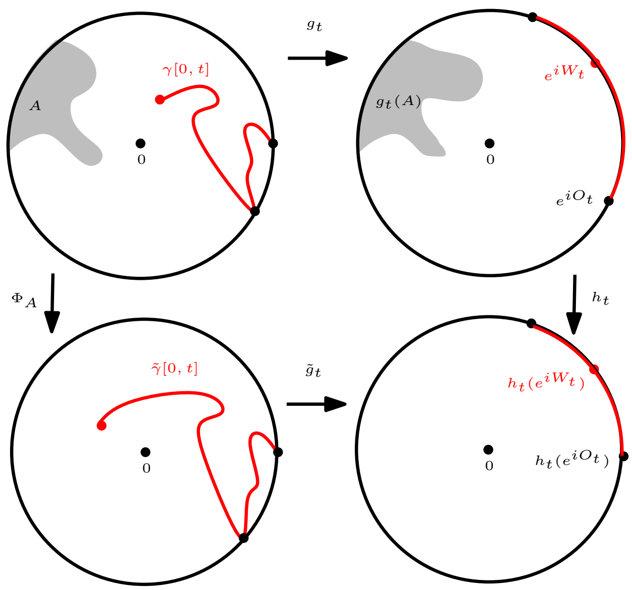


Fig. 5.4: Φ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ with $\Phi_A(0)=0, \Phi_A(1)=1$. g_t is the conformal map from $\mathbb{U}\setminus\gamma[0,t]$ onto $\mathbb{U}$ normalized at the origin. Define $\tilde{g}_t$ as the conformal map from $\mathbb{U}\setminus\tilde{\gamma}[0,t]$ onto $\mathbb{U}$ normalized at the origin and h_t as the conformal map from $\mathbb{U}\setminus g_t(A)$ onto $\mathbb{U}$ such that $h_t \circ g_t = \tilde{g}_t \circ \Phi_A$.

Proof. Define $\phi_t(z) = -i\log h_t(e^{iz})$ where log denotes the branch of the logarithm such that $-i\log h_t(e^{iW_t}) = W_t$. Then

$$|h_t'(e^{iW_t})| = \phi_t'(W_t), \quad |h_t'(e^{iV_t})| = \phi_t'(V_t), \quad \vartheta_t = (\phi_t(W_t) - \phi_t(V_t))/2.$$

To simplify the notations, we set $X_1 = \phi_t'(W_t), X_2 = \phi_t''(W_t), Y_1 = \phi_t'(V_t)$. By Itô formula, we have that

$$\begin{aligned}
d\phi_t(W_t) &= \sqrt{8/3}X_1 dB_t + \left(-\frac{5}{3}X_2 + \frac{\rho}{2}X_1\cot\theta_t\right)dt,\\
d\phi_t(V_t) &= -X_1^2\cot\vartheta_t dt,\\
d\phi_t'(W_t) &= \sqrt{8/3}X_2 dB_t + \left(\frac{\rho}{2}X_2\cot\theta_t + \frac{X_2^2}{2X_1} + \frac{X_1 - X_1^3}{6}\right)dt,\\
d\phi_t'(V_t) &= \left(-\frac{1}{2}X_1^2 Y_1\frac{1}{\sin^2\vartheta_t} + \frac{1}{2}Y_1\frac{1}{\sin^2\theta_t}\right)dt,\\
d\theta_t &= \frac{\sqrt{8/3}}{2}dB_t + \frac{\rho+2}{4}\cot\theta_t dt,\\
d\vartheta_t &= \frac{\sqrt{8/3}}{2}X_1 dB_t + \left(-\frac{5}{6}X_2 + \frac{1}{2}X_1^2\cot\vartheta_t + \frac{\rho}{4}X_1\cot\theta_t\right)dt.
\end{aligned}$$

Combine these, M is a local martingale. □

5.4 Relation between radial SLE and chordal SLE

Roughly speaking, chordal SLE is the limit of radial SLE when we let the interior target point go towards a boundary target point. Precisely, for $z \in \mathbb{H}$, suppose φ^z is the Mobius transformation from $\mathbb{U}$ onto $\mathbb{H}$ that sends 0 to z and 1 to 0. We define radial SLE in $\mathbb{H}$ from 0 to z as the image of radial SLE in $\mathbb{U}$ from 1 to 0 under φ^z. Then, as $y \to \infty$, radial SLE$_\kappa$ in $\mathbb{H}$ from 0 to iy will converge to chordal SLE$_\kappa$ (under an appropriate topology).

Proof. Fix $R > 0$, suppose $y > 0$ large. Let γ^{iy} be a radial SLE$_\kappa$ in $\mathbb{H}$ from 0 to iy and let γ be a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. Let τ_R be the first time that the curve exits $\mathbb{U}(0,R)$. Set $\rho = 6-\kappa$, and define

$$M_t(iy) = |g_t'(iy)|^{\rho(\rho+8-2\kappa)/(8\kappa)} (\Im g_t(iy))^{\rho^2/(8\kappa)} |g_t(iy) - W_t|^{\rho/\kappa}.$$

One can check that M is a local martingale under the law of γ (see [SW05, Theorem 6]). Moreover, the measure weighted by $M(iy)/M_0(iy)$ is the same as the law of γ^{iy} (after time-change). In particular, the Radon-Nikodym between the law of $\gamma^{iy}[0,\tau_R]$ and the law of $\gamma[0,\tau_R]$ is given by

$$M_{\tau_R}(iy)/M_0(iy) = |g_{\tau_R}'(iy)|^{\rho(\rho+8-2\kappa)/(8\kappa)} (\frac{\Im g_{\tau_R}(iy)}{y})^{\rho^2/(8\kappa)} (\frac{|g_{\tau_R}(iy) - W_{\tau_R}|}{y})^{\rho/\kappa}$$

which converges to 1 as $y \to \infty$. □

6 Lecture 6: Radial conformal restriction

6.1 Setup for radial restriction sample

Let Ω be the collection of compact subset K of $\bar{\mathbb{U}}$ such that

$$K \cap \partial \mathbb{U} = \{1\}, 0 \in K, K \text{ is connected and } \mathbb{U} \setminus K \text{ is connected.}$$

Recall $\mathscr{A}_r$ in Definition 1.2. Endow Ω with the σ-field generated by the events $[K \in \Omega : K \cap A = \emptyset]$ where $A \in \mathscr{A}_r$. Clearly, a probability measure $\mathbb{P}$ on Ω is characterized by the values of $\mathbb{P}[K \cap A = \emptyset]$ for $A \in \mathscr{A}_r$.

Definition 6.1. *A probability measure $\mathbb{P}$ on Ω is said to satisfy radial restriction property if the following is true:*

For any $A \in \mathscr{A}_r$, $\Phi_A(K)$ conditioned on $[K \cap A = \emptyset]$ has the same law as K.

Theorem 6.2. *1. (Characterization) A radial restriction measure is characterized by a pair of real numbers (α, β) such that, for every $A \in \mathscr{A}_r$,*

$$\mathbb{P}[K \cap A = \emptyset] = |\Phi_A'(0)|^{\alpha} \Phi_A'(1)^{\beta}. \tag{6.1}$$

We denote the corresponding radial restriction measure as $\mathbb{Q}(\alpha, \beta)$.

2. (Existence) The measure $\mathbb{Q}(\alpha, \beta)$ exists if and only if

$$\beta \geq 5/8, \alpha \leq \xi(\beta) = \frac{1}{48}((\sqrt{24\beta + 1} - 1)^2 - 4).$$

Homework: Suppose that K satisfies Equation (6.1) for any $A \in \mathscr{A}_r$, then K satisfies radial restriction property.

Remark 6.3. *We already know the existence of $\mathbb{Q}(5/48, 5/8)$ when K is radial $SLE_{8/3}$. Recall Theorem 2.17, if we take an independent Poisson point process with intensity $\alpha \mu^{loop}_{\mathbb{U},0}$, the "fill-in" of the union of the Poisson point process and radial $SLE_{8/3}$ would give $\mathbb{Q}(5/48 - \alpha, 5/8)$.*

Remark 6.4. *In Equation (6.1), we have that $|\Phi_A'(0)| \geq 1$ and $\Phi_A'(1) \leq 1$. Since β is positive, we have $\Phi_A'(1)^{\beta} \leq 1$. But α can be negative or positive, so that $|\Phi_A'(0)|^{\alpha}$ can be greater than 1. The product $|\Phi_A'(0)|^{\alpha} \Phi_A'(1)^{\beta}$ is always less than 1 which is guaranteed by the condition that $\alpha \leq \xi(\beta)$. (In fact, we always have $|\Phi_A'(0)| \Phi_A'(1)^2 \leq 1$.)*

Remark 6.5. *While the class of chordal restriction measures is characterized by one single parameter $\beta \geq 5/8$, the class of radial restriction measures involves the additional parameter α. This is due to the fact that the radial restriction property is in a sense weaker than the chordal one: the chordal restriction samples in $\mathbb{H}$ are scale-invariant, while the radial ones are not.*

6.2 Proof of Theorem 6.2. Characterization.

It is easier to carry out the calculation in the upper half-plane $\mathbb{H}$ instead of $\mathbb{U}$. Suppose that K satisfies radial restriction property in $\mathbb{H}$ with interior point i and the boundary point 0. In other words, K is the image of radial restriction sample in $\mathbb{U}$ under the conformal map $\varphi(z) = i(1-z)/(1+z)$. The proof consists of six steps. We will omit the proof of regularities and keep the proof that is related to the key idea.

Step 1. For any $x \in \mathbb{R} \setminus \{0\}$, the probability $\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon)]$ decays like ε^2 as ε goes to zero. And the limit

$$\lim_{\varepsilon \to 0} \mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset]$$

exists which we denoted as $\lambda(x)$. Furthermore $\lambda(x) \in (0,\infty)$.

Step 2. The function λ is continuous and differentiable for $x \in (-\infty,0) \cup (0,\infty)$.

Step 3. Fix $x, y \in \mathbb{R} \setminus \{0\}$. We estimate the probability of

$$\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset, K \cap \mathbb{U}(y,\delta) \neq \emptyset].$$

Clearly, it decays like $\varepsilon^2\delta^2$ as ε, δ go to zero. Denote $f_{x,\varepsilon}$ as the conformal map from $\mathbb{H} \setminus \mathbb{U}(x,\varepsilon)$ onto $\mathbb{H}$ that fixes 0 and i. In fact, we can write out the exact expression of $f_{x,\varepsilon}$: suppose $0 < \varepsilon < |x|$. Then

$$g_{x,\varepsilon}(z) := z + \frac{\varepsilon^2}{z-x}$$

is a conformal map from $\mathbb{H} \setminus \mathbb{U}(x,\varepsilon)$ onto $\mathbb{H}$. Define

$$f_{x,\varepsilon}(z) = b\frac{g_{x,\varepsilon}(z) - c}{b^2 + (c-a)(g_{x,\varepsilon}(z) - a)}$$

where $a = \Re(g_{x,\varepsilon}(i)), b = \Im(g_{x,\varepsilon}(i)), c = g_{x,\varepsilon}(0)$. Then $f_{x,\varepsilon}$ is the conformal map from $\mathbb{H} \setminus \mathbb{U}(x,\varepsilon)$ onto $\mathbb{H}$ that preserves 0 and i.

Lemma 6.6.

$$\lim_{\varepsilon \to 0}\lim_{\delta \to 0} \frac{1}{\varepsilon^2\delta^2}\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset, K \cap \mathbb{U}(y,\delta) \neq \emptyset] = \lambda(x)\lambda(y) - \lambda'(y)F(x,y) - 2\lambda(y)G(x,y)$$

where

$$F(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}(f_{x,\varepsilon}(y) - y) = \frac{1+x^2+y^2+xy}{x(1+x^2)} + \frac{1}{y-x},$$

$$G(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}(f'_{x,\varepsilon}(y) - 1) = \frac{x+2y}{x(1+x^2)} - \frac{1}{(y-x)^2}.$$

Proof. By radial restriction property, we have that

$$\begin{aligned}
&\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) = \emptyset, K \cap \mathbb{U}(y,\delta) \neq \emptyset] \\
=\ & \mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) = \emptyset] \times \mathbb{P}[K \cap \mathbb{U}(y,\delta) \neq \emptyset \,|\, K \cap \mathbb{U}(x,\varepsilon) = \emptyset] \\
=\ & \mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) = \emptyset] \times \mathbb{P}[K \cap f_{x,\varepsilon}(\mathbb{U}(y,\delta)) \neq \emptyset]
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\varepsilon \to 0}\lim_{\delta \to 0} \frac{1}{\varepsilon^2\delta^2}\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset, K \cap \mathbb{U}(y,\delta) \neq \emptyset] \\
=\ & \lim_{\varepsilon \to 0}\lim_{\delta \to 0} \frac{1}{\varepsilon^2\delta^2}\left(\mathbb{P}[K \cap \mathbb{U}(y,\delta) \neq \emptyset] - \mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) = \emptyset] \times \mathbb{P}[K \cap f_{x,\varepsilon}(\mathbb{U}(y,\delta)) \neq \emptyset]\right) \\
=\ & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left(\lambda(y) - \mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) = \emptyset]\lambda(f_{x,\varepsilon}(y))|f'_{x,\varepsilon}(y)|^2\right) \\
=\ & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left(\mathbb{P}[K \cap \mathbb{U}(x,\varepsilon) \neq \emptyset]\lambda(f_{x,\varepsilon}(y))|f'_{x,\varepsilon}(y)|^2 + \lambda(y) - \lambda(f_{x,\varepsilon}(y))|f'_{x,\varepsilon}(y)|^2\right) \\
=\ & \lambda(x)\lambda(y) - \lambda'(y)F(x,y) - 2\lambda(y)G(x,y).
\end{aligned}$$

□

Step 4. We are allowed to exchange the order in taking the limits in Lemma 6.6. In other words, we have that

$$\lambda'(y)F(x,y)+2\lambda(y)G(x,y)=\lambda'(x)F(y,x)+2\lambda(x)G(y,x). \tag{6.2}$$

We call this equation as **Commutation Relation**.

Step 5. Decide the expression of the function λ.

Lemma 6.7. *There exists two constants $c_0 \geq 0, c_2 \geq 0$ such that*

$$\lambda(x)=\frac{c_0+c_2x^2}{x^2(1+x^2)^2}.$$

Proof. In Commutation Relation (6.2), let $y \to x$, we obtain a differential equation for the function λ. To write it in a better way, define

$$P(x)=x^2(1+x^2)^2\lambda(x),$$

then the differential equation becomes

$$P'''(x)=0.$$

And we know that λ is an even function. Thus, there exist three constants c_0, c_1, c_2 such that

$$\lambda(x)=\frac{c_0+c_1x+c_2x^2}{x^2(1+x^2)^2}, \quad \text{for } x>0;$$

$$\lambda(x)=\frac{c_0-c_1x+c_2x^2}{x^2(1+x^2)^2}, \quad \text{for } x<0.$$

We plugin the expression of λ in Commutation Relation (6.2) and take $x>0>y$, then we get $c_1=0$. Since λ is positive, we have $c_0 \geq 0, c_2 \geq 0$. □

Step 6. Decide the relation between (α,β) and (c_0,c_2). Since there are only two-degree of freedom, when K satisfies radial restriction property, we must have that Equation (6.1) holds for some α,β. Note that

$$\mathbb{P}[K\cap \mathbb{U}(x,\varepsilon)\neq \emptyset]\sim \lambda(x)\varepsilon^2.$$

Compare it with

$$1-|f'_{x,\varepsilon}(i)|^{\alpha} f'_{x,\varepsilon}(0)^{\beta},$$

we have that

$$\alpha=(c_0-c_2)/4, \quad \beta=c_0/2.$$

6.3 Several basic observations of radial restriction property

Recall a result for Brownian loop: Theorem 2.17. Let $(l_j, j\in J)$ be a Poisson point process with intensity $c\mu^{loop}_{\mathbb{U},0}$ for some $c>0$. Set $\Sigma=\cup_j l_j$. Then we have that, for any $A\in\mathscr{A}_r$,

$$\mathbb{P}[\Sigma\cap A=\emptyset]=|\Phi'_A(0)|^{-c}.$$

Suppose K_0 is a radial restriction sample whose law is $\mathbb{Q}(\alpha_0,\beta_0)$. Take K as the "fill-in" of the union of Σ and K_0, then clearly, K has the law of $\mathbb{Q}(\alpha_0-c,\beta_0)$. Thus we derived the following lemma.

Lemma 6.8. *If the radial restriction measure exists for some (α_0,β_0), then $\mathbb{Q}(\alpha,\beta_0)$ exists for all $\alpha<\alpha_0$. Furthermore, almost surely for $\mathbb{Q}(\alpha,\beta_0)$, the origin is not on the boundary of K.*

In the next subsection, we will construct $\mathbb{Q}(\xi(\beta),\beta)$ for $\beta \geq 5/8$ and point out that if K has the law of $\mathbb{Q}(\xi(\beta),\beta)$, almost surely the origin is on the boundary of K. Thus, combine with Lemma 6.8, we could show that, when $\beta \geq 5/8$, $\mathbb{Q}(\alpha,\beta)$ exists if and only if $\alpha \leq \xi(\beta)$. Note that, radial $\mathrm{SLE}_{8/3}$ has the same law as $\mathbb{Q}(5/48,5/8)$ where $5/48 = \xi(5/8)$.

Another basic observation is that $\mathbb{Q}(\alpha,\beta)$ does not exist when $\beta < 5/8$. Suppose K^0 is a radial restriction sample with law $\mathbb{Q}(\alpha,\beta)$. For any interior point $z \in \mathbb{H}$, we define K^z as the image of K^0 under the Mobius transformation from $\mathbb{U}$ onto $\mathbb{H}$ such that sends 1 to 0 and 0 to z. Similar as the relation between radial SLE and chordal SLE in Subsection 5.4, if we let $z \to \infty$, K^z converges weakly toward some probability measure, and the limit measure satisfies chordal restriction property with exponent β, thus $\beta \geq 5/8$.

6.4 Construction of radial restriction measure $\mathbb{Q}(\xi(\beta),\beta)$ for $\beta > 5/8$

The construction of $\mathbb{Q}(\xi(\beta),\beta)$ is very similar to the construction of $\mathbb{P}(\beta)$.

Proposition 6.9. *Fix $\beta > 5/8$ and let*

$$\rho = \rho(\beta) = \frac{1}{3}(-8+2\sqrt{24\beta+1}).$$

Let γ^R be a radial $\mathrm{SLE}^L_{8/3}(\rho)$ in $\bar{\mathbb{U}}$ from 1 to 0. Given γ^R, let γ^L be an independent chordal $\mathrm{SLE}^R_{8/3}(\rho-2)$ in $\overline{\mathbb{U}\setminus\gamma^R}$ from 1^- to 0. Let K be the closure of the union of the domains between γ^L and γ^R. See Figure 6.1. Then the law of K is $\mathbb{Q}(\xi(\beta),\beta)$. In particular, the origin is almost surely on the boundary of K.

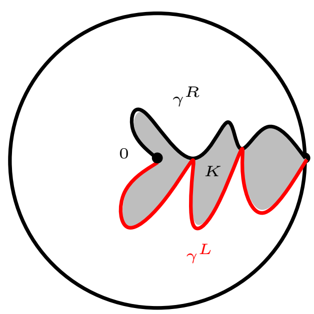


Fig. 6.1: γ^R is a radial $\mathrm{SLE}^L_{8/3}(\rho)$ in $\mathbb{U}$ from 1 to 0. Conditioned on γ^R, γ^L is a chordal $\mathrm{SLE}^R_{8/3}(\rho-2)$ in $\overline{\mathbb{U}\setminus\gamma^R}$ from 1^- to 0. K is the closure of the union of domains between the two curves.

Proof. Suppose $(g_t, t \geq 0)$ is the Loewner chain for γ^R. For any $A \in \mathscr{A}_r$, define $T = \inf\{t : \gamma^R(t) \in A\}$. For $t < T$, let h_t be the conformal map from $\mathbb{U}\setminus g_t(A)$ onto U normalized at the origin. From Proposition 5.7, we know that

$$M_t = |h_t'(0)|^{\alpha} \times |h_t'(e^{iW_t})|^{5/8} \times |h_t'(e^{iO_t})|^{\rho(3\rho+4)/32} \times (\frac{\sin\vartheta_t}{\sin\theta_t})^{3\rho/8}$$

is a local martingale, and

$$M_0 = |\Phi_A'(0)|^{\xi(\beta)}\Phi_A'(1)^{\beta}.$$

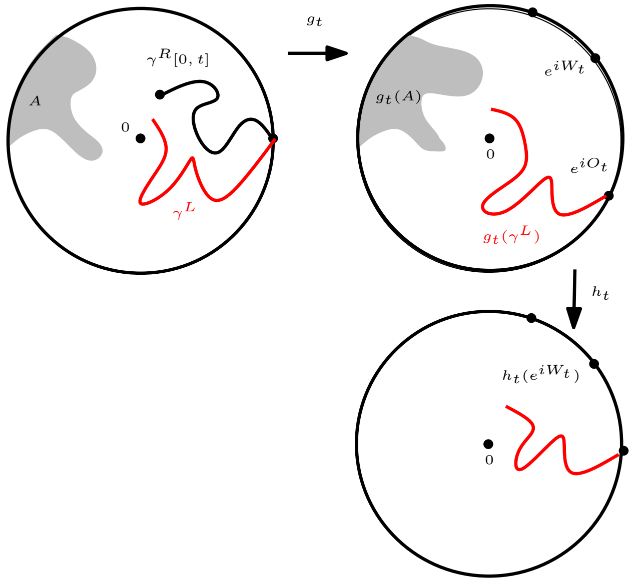


Fig. 6.2: g_t is the conformal map from $\mathbb{U} \setminus \gamma^R[0,t]$ onto $\mathbb{U}$ normalized at the origin. h_t is the conformal map from $\mathbb{U} \setminus g_t(A)$ onto $\mathbb{U}$ normalized at the origin.

See Figure 6.2. If $T < \infty$,

$$\lim_{t \to T} h_t'(e^{iW_t}) = 0, \quad \text{and } \lim_{t \to T} M_t = 0.$$

If $T = \infty$, as $t \to \infty$

$$|h_t'(0)| \to 1, \quad |h_t'(e^{iW_t})| \to 1, \quad \frac{\sin \theta_t}{\sin \vartheta_t} \to 1,$$

and

$$|h_t'(e^{iO_t})|^{\rho(3\rho+4)/32} \to \mathbb{P}[\gamma^L \cap A = \emptyset \,|\, \gamma^R].$$

Thus,

$$\mathbb{P}[K \cap A = \emptyset] = \mathbb{E}[1_{T=\infty}\mathbb{E}[1_{K\cap A=\emptyset}] \,|\, \gamma^R] = \mathbb{E}[M_T] = M_0.$$

□

6.5 Whole-plane intersection exponents $\xi(\beta_1, ..., \beta_p)$

Recall that

$$\tilde{\xi}(\beta_1, ..., \beta_p) = \frac{1}{24}\left((\sqrt{24\beta_1 + 1} + \cdots + \sqrt{24\beta_p + 1} - (p-1))^2 - 1\right),$$

$$\hat{\tilde{\xi}}(\beta_1, ..., \beta_p) = \tilde{\xi}(\beta_1, ..., \beta_p) - \beta_1 - \cdots - \beta_p,$$

$$\xi(\beta_1, ..., \beta_p) = \frac{1}{48}\left((\sqrt{24\beta_1 + 1} + \cdots + \sqrt{24\beta_p + 1} - p)^2 - 4\right).$$

For $x \in \mathbb{R}$ and a subset $K \subset \bar{\mathbb{U}}$, denote

$$e^{ix}K = \{e^{ix}z : z \in K\}.$$

For $r \in (0,1)$, denote the annulus

$$\mathbb{A}_r = \mathbb{U} \setminus \bar{\mathbb{U}}(0,r).$$

Proposition 6.10. *Fix $\beta_1,...,\beta_p \geq 5/8$. Suppose $K_1,...,K_p$ are p independent radial restriction samples whose laws are $\mathbb{Q}(\xi(\beta_1),\beta_1)$,..., $\mathbb{Q}(\xi(\beta_p),\beta_p)$ respectively. Let $r > 0$, $\varepsilon > 0$ small. Set $x_j = j\varepsilon$ for $j = 1,...,p$. Then*

$$\begin{aligned}&\mathbb{P}[(e^{ix_{j_1}}K_{j_1}\cap\mathbb{A}_r)\cap(e^{ix_{j_2}}K_{j_2}\cap\mathbb{A}_r)=\emptyset, 1\leq j_1<j_2\leq p]\\ \approx\ &\varepsilon^{\hat{\xi}(\beta_1,...,\beta_p)}r^{\xi(\beta_1,...,\beta_p)-\xi(\beta_1)-\cdots-\xi(\beta_p)} \quad \text{as } \varepsilon, r\to 0.\end{aligned}$$

In the following theorem, we will consider the law of $K_1,...,K_p$ conditioned on "non-intersection". Since the event of "non-intersection" has zero probability, we need to explain the precise meaning: the conditioned law would be obtained through a limiting procedure: first consider the law of $K_1,...,K_p$ conditioned on

$$[(e^{ix_{j_1}}K_{j_1}\cap\mathbb{A}_r)\cap(e^{ix_{j_2}}K_{j_2}\cap\mathbb{A}_r)=\emptyset, 1\leq j_1<j_2\leq p]$$

and then let $r\to 0$ and $\varepsilon\to 0$.

Theorem 6.11. *Fix $\beta_1,...,\beta_p \geq 5/8$. Suppose $K_1,...,K_p$ are p independent radial restriction samples whose laws are $\mathbb{Q}(\xi(\beta_1),\beta_1)$,..., $\mathbb{Q}(\xi(\beta_p),\beta_p)$ respectively. Then the "fill-in" of the union of these p sets conditioned on "non-intersection" has the same law as radial restriction sample with law*

$$\mathbb{Q}(\xi(\beta_1,...,\beta_p),\tilde{\xi}(\beta_1,...,\beta_p)).$$

We only need to show the results for $p = 2$ and other p can be proved by induction. When $p = 2$, Proposition 6.10 is a direct consequence of the following lemma.

Lemma 6.12. *Let K be a radial restriction sample with exponents (α,β). Let $r > 0$, $\varepsilon > 0$ be small. Suppose γ is an independent radial $SLE^L_{8/3}(\rho)$ process. Then we have*

$$\mathbb{P}[\gamma[0,t]\cap(e^{i\varepsilon}K)=\emptyset]\approx\varepsilon^{\frac{3}{16}\bar{\rho}(\rho+2)}r^{\bar{q}-q-\alpha} \quad \text{as } \varepsilon, r\to 0$$

where $r = e^{-t}$, and

$$\bar{\rho}=\frac{2}{3}(\sqrt{24\beta+1}-1),$$

$$q=\frac{3}{64}\rho(\rho+4), \quad \bar{q}=\frac{3}{64}(\bar{\rho}+\rho)(\bar{\rho}+\rho+4).$$

Note that, if $\beta_1=\beta$, $\beta_2=(3\rho^2+16\rho+20)/32, \alpha=\xi(\beta_1)$, then

$$\frac{3}{16}\bar{\rho}(\rho+2)=\hat{\xi}(\beta_1,\beta_2), \quad \bar{q}-q-\alpha=\xi(\beta_1,\beta_2)-\xi(\beta_1)-\xi(\beta_2).$$

Proof. Let $(g_t, t\geq 0)$ be the Loewner chain for γ and (O_t,W_t) be the solution to the SDE. Precisely,

$$\partial_t g_t(z)=g_t(z)\frac{e^{iW_t}+g_t(z)}{e^{iW_t}-g_t(z)}, \quad g_0(z)=z;$$

$$dW_t=\sqrt{\kappa}dB_t+\frac{\rho}{2}\cot(\frac{W_t-O_t}{2})dt, \quad dO_t=-\cot(\frac{W_t-O_t}{2})dt, \quad W_0=0, O_0=2\pi-.$$

Given $\gamma[0,t]$, since K satisfies radial restriction property, we have that

$$\mathbb{P}[\gamma[0,t]\cap(e^{i\varepsilon}K)=\emptyset\,|\,\gamma[0,t]]=g_t'(e^{i\varepsilon})^{\beta}e^{t\alpha}.$$

Define

$$M_t = e^{t(\bar{q}-q)} g_t'(e^{i\varepsilon})^{\beta} |g_t(e^{i\varepsilon}) - e^{iW_t}|^{3\bar{\rho}/8} |g_t(e^{i\varepsilon}) - e^{iO_t}|^{3\rho\bar{\rho}/16}.$$

One can check that M is a local martingale. Thus we have

$$\begin{aligned}
&\mathbb{P}[\gamma[0,t] \cap (e^{i\varepsilon}K) = \emptyset] \\
&= \mathbb{E}[g_t'(e^{i\varepsilon})^{\beta} e^{t\alpha}] \\
&= e^{t(\alpha-\bar{q}+q)} \mathbb{E}[e^{t(\bar{q}-q)} g_t'(e^{i\varepsilon})^{\beta}] \\
&\approx r^{\bar{q}-q-\alpha} \mathbb{E}[M_t] = r^{\bar{q}-q-\alpha} M_0.
\end{aligned}$$

□

Proof of Theorem 6.11. Assume $p = 2$. For any $A \in \mathscr{A}_r$, we need to estimate the following probability

$$\mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset \,|\, (e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset].$$

Since K_i satisfies radial restriction property, conditioned on $[K_i \cap A = \emptyset]$, the conditional law of $\Phi_A(K_i)$ has the same law as K_i for $i = 1, 2$. Thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0, r\to 0} \mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset \,|\, (e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset] \\
&= \lim_{\varepsilon\to 0, r\to 0} \frac{\mathbb{P}[K_1 \cap A = \emptyset, K_2 \cap A = \emptyset, (e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset]}{\mathbb{P}[(e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset]} \\
&= \lim_{\varepsilon\to 0, r\to 0} |\Phi_A'(0)|^{\xi(\beta_1)+\xi(\beta_2)} \Phi_A'(1)^{\beta_1+\beta_2} \frac{\mathbb{P}[(e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset \,|\, K_1 \cap A = \emptyset, K_2 \cap A = \emptyset]}{\mathbb{P}[(e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset]} \\
&= \lim_{\varepsilon\to 0, r\to 0} |\Phi_A'(0)|^{\xi(\beta_1)+\xi(\beta_2)} \Phi_A'(1)^{\beta_1+\beta_2} \frac{\mathbb{P}[(\Phi_A(e^{i\varepsilon})K_1 \cap \Phi_A(\mathbb{A}_r \setminus A) \cap (\Phi_A(e^{i2\varepsilon})K_2 \cap \Phi_A(\mathbb{A}_r \setminus A)) = \emptyset]}{\mathbb{P}[(e^{i\varepsilon}K_1 \cap \mathbb{A}_r) \cap (e^{i2\varepsilon}K_2 \cap \mathbb{A}_r) = \emptyset]} \\
&= |\Phi_A'(0)|^{\xi(\beta_1)+\xi(\beta_2)} \Phi_A'(1)^{\beta_1+\beta_2} \Phi_A'(1)^{\tilde{\xi}(\beta_1,\beta_2)} |\Phi_A'(0)|^{\xi(\beta_1,\beta_2)-\xi(\beta_1)-\xi(\beta_2)} \\
&= |\Phi_A'(0)|^{\xi(\beta_1,\beta_2)} \Phi_A'(1)^{\tilde{\xi}(\beta_1,\beta_2)}.
\end{aligned}$$

□

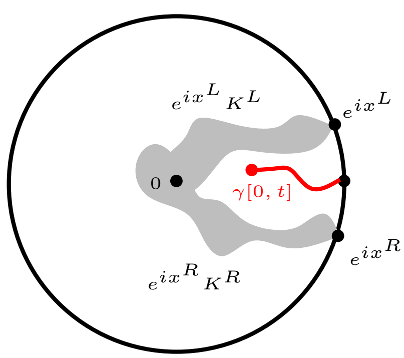


Fig. 6.3: K^L is a radial restriction sample with exponents (α^L, β^L), and K^R is a radial restriction sample with exponents (α^R, β^R). Let γ be a radial $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ in $\mathbb{U}$ from 1 to 0 with force points $(x^L; x^R)$ where $\kappa > 0, \rho^L > -2, \rho^R > -2$, and $x^L < 0 < x^R$.

6.6 Related calculation

Radial SLE$_\kappa(\rho^L;\rho^R)$ process

Suppose $\kappa>0, \rho^L>-2, \rho^R>-2$ and $0<x^R<x^L<2\pi$. Radial SLE$_\kappa(\rho^L;\rho^R)$ process with force points $(e^{ix^L};e^{ix^R})$ is the Loewner chain driven by W which is the solution to the following SDE:

$$dW_t=\sqrt{\kappa}dB_t+\frac{\rho^L}{2}\cot(\frac{W_t-O_t^L}{2})+\frac{\rho^R}{2}\cot(\frac{W_t-O_t^R}{2})dt,$$

$$dO_t^L=-\cot(\frac{W_t-O_t^L}{2})dt,\quad dO_t^R=-\cot(\frac{W_t-O_t^R}{2})dt,\quad W_0=0, O_0^L=x^L, O_0^R=x^R.$$

There exists piecewise unique solution to the above SDE. And there exists almost surely a continuous curve γ in $\bar{\mathbb{U}}$ from 1 to 0 associated to the radial SLE$_\kappa(\rho^L;\rho^R)$ process with force points $(e^{ix^L};e^{ix^R})$. Note that, for small time t when e^{ix^L}, e^{ix^R} are not swallowed by K_t, e^{ix^L} (resp. e^{ix^R}) is the preimage of $e^{iO_t^L}$ (resp. $e^{iO_t^R}$) under g_t.

It is worthwhile to point out the relation between radial SLE$_\kappa(\rho^L;\rho^R)$ processes with different ρ's. Fix $\kappa>0$, $0<x^R<x^L<2\pi$. Let $\rho^L>-2$, $\rho^R>-2$, $\tilde{\rho}^L>-2$, $\tilde{\rho}^R>-2$. Set

$$q=\frac{1}{8\kappa}(\rho^L+\rho^R)(\rho^L+\rho^R+4),\quad \tilde{q}=\frac{1}{8\kappa}(\tilde{\rho}^L+\tilde{\rho}^R)(\tilde{\rho}^L+\tilde{\rho}^R+4).$$

Define

$$\begin{aligned}M_t=&\, e^{t(\tilde{q}-q)}g_t'(e^{ix^L})^{(\tilde{\rho}^L-\rho^L)(\tilde{\rho}^L+\rho^L+4-\kappa)/(4\kappa)}g_t'(e^{ix^R})^{(\tilde{\rho}^R-\rho^R)(\tilde{\rho}^R+\rho^R+4-\kappa)/(4\kappa)}\\ &\times|g_t(e^{ix^L})-e^{iW_t}|^{(\tilde{\rho}^L-\rho^L)/\kappa}|g_t(e^{ix^R})-e^{iW_t}|^{(\tilde{\rho}^R-\rho^R)/\kappa}\\ &\times|g_t(e^{ix^L})-g_t(e^{ix^R})|^{(\tilde{\rho}^L\tilde{\rho}^R-\rho^L\rho^R)/(2\kappa)}.\end{aligned}$$

Then M is a local martingale under the measure of radial SLE$_\kappa(\rho^L;\rho^R)$ process (see [SW05, Equation (9)]). Moreover, the measure weighted by M/M_0 (with an appropriate stopping time) is the same as the law of radial SLE$_\kappa(\tilde{\rho}^L;\tilde{\rho}^R)$ process.

General calculation related to Lemma 6.12

Suppose K^L is a radial restriction sample with exponents (α^L,β^L), and K^R is a radial restriction sample with exponents (α^R,β^R). Let γ be a radial SLE$_\kappa(\rho^L;\rho^R)$ in $\mathbb{U}$ from 1 to 0 with force points $(e^{ix^L};e^{ix^R})$ where $\kappa>0, \rho^L>-2, \rho^R>-2$, and $x^L<0<x^R$. See Figure 6.3. Suppose K^L, K^R, γ are independent and let $\bar{\rho}^L>\rho^L, \bar{\rho}^R>\rho^R$ be the solutions to the equations

$$\beta^L=\frac{1}{4\kappa}(\bar{\rho}^L-\rho^L)(\bar{\rho}^L+\rho^L+4-\kappa),\quad \beta^R=\frac{1}{4\kappa}(\bar{\rho}^R-\rho^R)(\bar{\rho}^R+\rho^R+4-\kappa).$$

And set

$$q=\frac{1}{8\kappa}(\rho^L+\rho^R)(\rho^L+\rho^R+4),\quad \bar{q}=\frac{1}{8\kappa}(\bar{\rho}^L+\bar{\rho}^R)(\bar{\rho}^L+\bar{\rho}^R+4).$$

Then,

$$\begin{aligned}&\mathbb{P}[\gamma[0,t]\cap(e^{ix^L}K^L)=\emptyset, \gamma[0,t]\cap(e^{ix^R}K^R)=\emptyset]\\ &\approx\ r^{\bar{q}-q-\alpha^L-\alpha^R}|x^L|^{(\bar{\rho}^L-\rho^L)/\kappa}|x^R|^{(\bar{\rho}^R-\rho^R)/\kappa}|x^R-x^L|^{(\bar{\rho}^L\bar{\rho}^R-\rho^L\rho^R)/(2\kappa)}\quad \text{as } x^L, x^R, r\to 0\end{aligned}$$

where $r=e^{-t}$.

Proof. Let $(g_t, t \geq 0)$ be the Loewner chain for γ. Given $\gamma[0,t]$, we have that

$$\mathbb{P}[(e^{ix^L}K^L) \cap \gamma[0,t] = \emptyset \,|\, \gamma[0,t]] = g_t'(e^{ix^L})^{\beta^L} e^{t\alpha^L}, \quad \mathbb{P}[(e^{ix^R}K^R) \cap \gamma[0,t] = \emptyset \,|\, \gamma[0,t]] = g_t'(e^{ix^R})^{\beta^R} e^{t\alpha^R}.$$

Note that

$$\begin{aligned} M_t =& e^{t(\bar{q}-q)} g_t'(e^{ix^L})^{\beta^L} g_t'(e^{ix^R})^{\beta^R} \\ & \times |g_t(e^{ix^L}) - e^{iW_t}|^{(\bar{\rho}^L - \rho^L)/\kappa} |g_t(e^{ix^R}) - e^{iW_t}|^{(\bar{\rho}^R - \rho^R)/\kappa} |g_t(e^{ix^R}) - g_t(e^{ix^L})|^{(\bar{\rho}^L\bar{\rho}^R - \rho^L\rho^R)/(2\kappa)} \end{aligned}$$

is a local martingale. Thus,

$$\begin{aligned} & \mathbb{P}[\gamma[0,t] \cap (e^{ix^L}K^L) = \emptyset, \gamma[0,t] \cap (e^{ix^R}K^R) = \emptyset] \\ = \;& \mathbb{E}[e^{t(\alpha^L+\alpha^R)} g_t'(e^{ix^L})^{\beta^L} g_t'(e^{ix^R})^{\beta^R}] \\ = \;& e^{t(\alpha^L+\alpha^R-\bar{q}+q)} \mathbb{E}[e^{t(\bar{q}-q)} g_t'(e^{ix^L})^{\beta^L} g_t'(e^{ix^R})^{\beta^R}] \\ \approx \;& r^{\bar{q}-q-\alpha^L-\alpha^R} \mathbb{E}[M_t] = r^{\bar{q}-q-\alpha^L-\alpha^R} M_0. \end{aligned}$$

□

References

[Bil99] Patrick Billingsley. *Convergence of Probability Measures*. 1999.

[DK88] Bertrand Duplantier and Kyung-Hoon Kwon. Conformal invariance and intersections of random walks. *Phys. Rev. Lett.*, 61:2514–2517, Nov 1988.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[LSW01a] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.

[LSW01b] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. II. Plane exponents. *Acta Math.*, 187(2):275–308, 2001.

[LSW02] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. III. Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(1):109–123, 2002.

[LSW03] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955 (electronic), 2003.

[LW99] Gregory F. Lawler and Wendelin Werner. Intersection exponent for planar brownian motion. *The Annals of Probability*, 27(4):1601–1642, 1999.

[LW00] Gregory F. Lawler and Wendelin Werner. Universality for conformally invariant intersection exponents. *J. Eur. Math. Soc. (JEMS)*, 2(4):291–328, 2000.

[LW04] Gregory F. Lawler and Wendelin Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[SW05] Oded Schramm and David B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669 (electronic), 2005.

[SW12] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Ann. of Math. (2)*, 176(3):1827–1917, 2012.

[Wer04] Wendelin Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 107–195. Springer, Berlin, 2004.

[Wer05] Wendelin Werner. Conformal restriction and related questions. *Probab. Surv.*, 2:145–190, 2005.

[Wer07] Wendelin Werner. Lectures on two-dimensional critical percolation. 2007.

[Wer08] Wendelin Werner. The conformally invariant measure on self-avoiding loops. *J. Amer. Math. Soc.*, 21(1):137–169, 2008.

[Wu13] Hao Wu. Conformal restriction: the radial case. 2013.