Lecture Notes at Tsinghua 2014
Introduction to SCV
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1. Holomorphic Functions

The following standard notation will be used: $\mathbb{R}$ denotes the real numbers; $\mathbb{C}$ denotes the complex numbers; $(x_1, \ldots, x^N)$ denotes an element of $\mathbb{R}^N$; and

$$(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \approx (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

denotes an element of $\mathbb{C}^n \approx \mathbb{R}^{2n}$. The differential operators on $\mathbb{C}^n$ defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}),\ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}),\ j = 1, \ldots, n,$$

will prove useful. We have

$$\frac{\partial}{\partial z_j} z_k = \frac{\partial}{\partial \overline{z}_j} \overline{z}_k = \delta_{jk},\ \ \frac{\partial}{\partial z_j} \overline{z}_k = \frac{\partial}{\partial z_j} \overline{z}_k = 0.$$

Definition 1.1. Let $D \subset \mathbb{C}^n$ be open. A function $f : D \to \mathbb{C}$ is said to be *holomorphic* if $f \in C^1(D)$ and if

$$\frac{\partial f}{\partial \overline{z}_j}(z) = 0 \quad \text{for } 1 \le j \le n \text{ and } z \in D.$$

The family of holomorphic functions on D is denoted by $\mathscr{O}(D)$. □

Let $f = u + iv$. Then

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}_j} &= \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})(u + iv) \\ &= \frac{1}{2}(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j}) + \frac{i}{2}(\frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j}) \end{aligned}$$

Thus $\partial f / \partial \overline{z}_j = 0$ means that f is holomorphic (or analytic) in z_j. So we have the following equivalent definition.

Definition 1.2. Let $D \subset \mathbb{C}^n$ be open. A function $f : D \to \mathbb{C}$ is said to be *holomorphic* if $f \in C^1(D)$ and if f is holomorphic in each variable separately. □

The following theorem is discovered by F. Hartogs in 1906.

Theorem 1.3. *(Hartogs (Friedrich Moritz Hartogs, 1874–1943) Theorem, 1906) Each function $f : D \to \mathbb{C}$ which is holomorphic in each variable separately is holomorphic.* □

Exercise 1.4. Prove that each function $f : D \to \mathbb{C}$ which is holomorphic in each variable separately is a Borel function. □

Hint: see Rudin, Real and complex analysis, 3rd ed, page 176.

If $a \in \mathbb{C}^n$, $\rho > 0$, $r \in \mathbb{R}^n_+$, then define the open ball

$$B_n(a, \rho) = \{z \in \mathbb{C}^n : |z - a| < \rho\}$$

and the open polydisc

$$\Delta^n(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j,\ j = 1, \ldots, n\}.$$

Let $\overline{B}_n(a, r)$ and $\overline{\Delta}^n(a, r)$ denote their closures respectively. The set

$$T^n(a, r) := \{z \in \mathbb{C}^n : |z_j - a_j| = r_j,\ j = 1, \ldots, n\}$$

is called the *distinguished boundary* of $\Delta^n(a, r)$. We write $\Delta^1(a, r) = \Delta(a, r)$ and $\Delta(0, 1) = \Delta$.

Let D be an open set in $\mathbb{C}^n$ and let $\overline{\Delta}^n(a, r) \subset D$.

Suppose that $f : D \to \mathbb{C}$ is holomorphic in z_1. Then

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a_1| = r_1} \frac{f(\zeta_1, z_2, \ldots, z_n)\, d\zeta_1}{\zeta_1 - z_1}, \quad z \in \Delta^n(a, r).$$

Exercise 1.5. Suppose that $f : D \to \mathbb{C}$ is holomorphic in each variable separately and that f is locally bounded. Use the above equation to prove that f is locally Lipschitz in each variable and hence f is continuous. □

Theorem 1.6. *(Cauchy Integral Formula) Suppose that $f : D \to \mathbb{C}$ is holomorphic in each variable separately and that f is continuous. Then*

$$f(z) = (2\pi i)^{-n} \int_{T^n(a,r)} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}, \quad z \in \Delta^n(a, r).$$

Proof. Since f is holomorphic in each variable separately, we have

$$f(z) = (2\pi i)^{-n} \int_{|\zeta_1 - a_1| = r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots \int_{|\zeta_n - a_n| = r_n} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_n}{\zeta_n - z_n},$$

for $z \in \Delta^n(a, r)$. Since f is continuous, the iterated integral in the above equation equals the integral in the statement, by Fubini's Theorem. □

Exercise 1.7. Prove Osgood's Theorem: if $f : D \to \mathbb{C}$ is holomorphic in each variable separately and if f is continuous then f is holomorphic. (Hint: use the above equation to prove that f is C^1.) □

Sometimes it is convenient to write $z = (z_1, \ldots, z_n) = (z', z_n)$, where $z' = (z_1, \ldots, z_{n-1})$. Similarly, $(r_1, \ldots, r_n) = (r', z_n)$. The crucial step in Hartogs' proof of his theorem is the following lemma.

Lemma 1.8. *(Hartogs Lemma) Suppose that $f : D \to \mathbb{C}$ is holomorphic in z' and in z_n separately. Then f is holomorphic on some nonempty open subset of D.*

Proof. Choose a closed polydisc $\overline{\Delta}^n(a, r)$ in D. Let

$$Q_j = \{z_n \in \overline{\Delta}(a_n, r_n) : |f(z', z_n)| \leq j \ \ \forall z' \in \overline{\Delta}^{n-1}(a', r')\}.$$

Then each Q_j is closed since it is the intersection of a family of closed sets:

$$Q_j = \cap_{z' \in \overline{\Delta}^{n-1}(a', r')} \{z_n \in \overline{\Delta}(a_n, r_n) : |f(z', z_n)| \leq j\}.$$

For each $z_n \in \overline{\Delta}(a_n, r_n)$, the function $f(\cdot, z_n)$ is bounded on $\overline{\Delta}^{n-1}(a', r')$. This means that z_n belongs to some Q_j. Thus $\overline{\Delta}(a_n, r_n) = \cup Q_j$. By Bair Category Theorem, some Q_j contains an open set $U \subset \Delta(a_n, r_n)$. Thus $|f(z', z_n)| \leq j$ on $\Delta^{n-1}(a', r') \times U$. The function f is continuous on $\Delta^{n-1}(a', r') \times U$ by Exercise 1.5, and holomorphic on $\Delta^{n-1}(a', r') \times U$ by Exercise 1.7. □

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, define $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and

$$D^\alpha = (\frac{\partial}{\partial z_1})^{\alpha_1} \cdots (\frac{\partial}{\partial z_n})^{\alpha_n}.$$

Theorem 1.9. *Let $f \in \mathscr{O}(D)$ and $\overline{\Delta}^n(a, r) \subset D$. Then, for $z \in \Delta^n(a, r)$ and $\alpha \in \mathbb{N}^n$,*

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{T^n(a,r)} \frac{f(\zeta)\, d\zeta_1 \dots d\zeta_n}{(\zeta - z)^{\alpha+1}}.$$

Hence each holomorphic function is C^∞.

Proof. Use Cauchy integral formula and differentiate under the integral sign. □

Exercise 1.10. Prove the Cauchy estimates: if $f \in \mathscr{O}(D)$ and $\overline{\Delta}^n(a, r) \subset D$, then

$$|D^\alpha f(a)| \leq \frac{\alpha!}{r^\alpha} |f|_{T^n(a,r)}.$$

Exercise 1.11. State and prove Liouville Theorem for several variables.

Exercise 1.12. State and prove the identity theorem for several variables.

Exercise 1.13. State and prove the maximum modulus theorem for several variables.

Let D be a domain (connected open set). Let $A(D)$ denote the set of functions which are holomorphic in D and continuous to the boundary ∂D.

Exercise 1.14. Let $f \in A(\Delta^n(0, r))$ and let ζ_n be a point on the circle $\{z_n \in \mathbb{C} : |z_n| = r_n\}$. Let $f_{\zeta_n} : \Delta^{n-1}(0, r') \to \mathbb{C}$ be defined by $f_{\zeta_n}(z') = f(z', \zeta_n)$. Prove that f_{ζ_n} is holomorphic in $\Delta^{n-1}(0, r')$.

Notation: $|g|_Q = \sup_{w \in Q} |g(w)|$.

Exercise 1.15. Let $f \in A(\Delta^n(0, r))$. Prove that $|f|_{\overline{\Delta}^n(0,r)} = |f|_{T^n(0,r)}$.

Exercise 1.16. Let S be a closed subset of the boundary of $\Delta^n(0, r))$ such that $|f|_{\overline{\Delta}^n(0,r)} = |f|_S$ for each $f \in A(\Delta^n(0, r)$. Prove that $S \supset T^n(0, r)$.

Exercise 1.17. Let $D \subset \mathbb{C}^n$ be a domain and let $b \in \mathbb{R}^n$ with $Q := \{z \in D : \Re z = b\} \neq \emptyset$. Prove that if $f \in \mathscr{O}(D)$ and if $f|_Q = 0$ then $f = 0$.

For $a, b \in (0, 1)$, let

$$U = \{(z_1, z_2) : |z_1| < a, |z_2| < 1\} \cup \{(z_1, z_2) : |z_1| < 1, b < |z_2| < 1\}.$$

Theorem 1.18. *(Hartogs' Extension Theorem) Each holomorphic function on U extends to a holomorphic function on $\Delta^2(0, 1)$.*

Proof. Let $f \in \mathscr{O}(U)$. Choose $c \in (b, 1)$ and define $g : \Delta^2(0, (1, c)) \to \mathbb{C}$ by

$$g(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta|=c} \frac{f(z_1, \zeta)\, d\zeta}{\zeta - z_2}.$$

Then g is holomorphic on $V := \Delta^2(0, (1, c))$ and coincides with f on $\Delta^2(0, (a, c))$. By the identity theorem $g = f$ on $U \cap V$. □

Exercise 1.19. Let $U = \Delta^n(0, 1) \setminus \overline{\Delta}^n((1, 0, \dots, 0), 1/2)$. Prove that each $f \in \mathscr{O}(U)$ extends holomorphically to a function on $\Delta^n(0, 1)$.

1.20. Digression: Newton's identities.

Let $x_1, \dots, x_n$ be variables. Let

$$\begin{aligned}
s_k &= \sum_{j=1}^n x_j^k, \quad k = 0, 1, 2, \dots, \\
\sigma_0 &= 1, \\
\sigma_k &= \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k}, \quad k = 1, \dots, n, \\
\sigma_k &= 0, \quad k > n.
\end{aligned}$$

Newton's identities:

$$k\sigma_k = s_1\sigma_{k-1} - s_2\sigma_{k-2} + \cdots + (-1)^{k-1} s_k \sigma_0, \quad k \ge 1.$$

Proof. For $1 \le j \le k$, let

$$Q_{kj} = \sum_{\substack{\ell \ne m_1, \dots, m_{k-j} \\ m_1 < \cdots < m_{k-j}}} x_\ell^j x_{m_1} \cdots x_{m_{k-j}}.$$

Set $Q_{k,k+1} = 0$. Then (Exercise: prove the following identities.)

$$s_j \sigma_{k-j} = Q_{kj} + Q_{k,j+1}, \quad 1 \le j \le k.$$

Taking the telescoping sum of the equations we obtain

$$\sum_{j=1}^k (-1)^{j-1} s_j \sigma_{k-j} = Q_{k1} = k\sigma_k.$$

□

By Newton's identities, σ_j are polynomials of s_k, and s_k are polynomials of σ_j.

We say f is a holomorphic function at 0 in $\mathbb{C}^n$, or f is holomorphic at a, if $f \in \mathscr{O}(B_n(a, \delta))$ for some $\delta = \delta_f > 0$. The set of holomorphic functions at a is denoted by $\mathscr{O}_{n,a}$. The functions f, g are considered the same element of $\mathscr{O}_{n,a}$ if $f = g$ on $B_n(a, \delta)$ for some $\delta > 0$. The set $\mathscr{O}_{n,a}$ is a ring. Set $\mathscr{O}_{n,0} = \mathscr{O}_n$. Let $f \in \mathscr{O}_{n,a}$, $a = (a', a_n)$. We say f is z_n-*regular* if the function $g(z_n) := f(a', z_n) - f(a)$ is not identically 0.

A *Weierstrass polynomial* in z_n of degree k is a polynomial of the form

$$p(z', z_n) = z_n^k + a_{k-1}(z') z_n^{k-1} + \cdots + a_0(z'),$$

where $a_0, \dots, a_{k-1}$ are holomorphic functions at 0 with $a_j(0) = 0$, $j = 0, \dots, k-1$.

Theorem 1.21. *(Weierstrass Preparation Theorem) Let f be holomorphic at 0 in $\mathbb{C}^n$. Suppose that $f(0) = 0$ and f is z_n-regular. Then there is a unique factorization*

$$f = p \cdot u,$$

where p is a Weierstrass polynomial in z_n, and u is holomorphic at 0 in $\mathbb{C}^n$ with $u(0) \neq 0$.

Proof. The uniqueness is clear. There is a $\delta = (\delta', \delta_n) \in \mathbb{R}^n_+$ such that f is holomorphic on $\overline{\Delta}^n(0, \delta)$,

$$f(0', z_n) \neq 0 \text{ if } 0 < |z_n| \leq \delta_n,$$

and

$$f(z', z_n) \neq 0, \text{ if } z' \in \overline{\Delta}^n(0', \delta') \text{ and } |z_n| = \delta_n.$$

For $z' \in \overline{\Delta}^n(0', \delta')$, consider the curve $\gamma_{z'}(t) = f(z', \delta_n e^{it})$, $0 \leq t \leq 2\pi$. The winding number k of $\gamma_{0'}$ equals the multiplicity of the function $g(\lambda) := f(0', \lambda)$ at 0, so $k > 0$. Then each $\gamma_{z'}$ has winding number k, since the curves form a continuous family. It follows that for each $z' \in \overline{\Delta}^n(0', \text{’})$, the function $f(z', \cdot)$ has exactly k zeros (counting multiplicity) in $\{|z_n| < \delta_n\}$. Let $\varphi_j(z')$, $j = 1, \ldots, k$, denote those zeros and put

$$p(z', z_n) = \Pi_j (z_n - \varphi_j(z')) := z_n^k + a_{k-1}(z') z_n^{k-1} + \cdots + a_0(z').$$

Since $\varphi_j(0) = 0$, we see that $a_j(0) = 0$. The function $u := f/p$ is well-defined and holomorphic in z_n. Let $S_m(z') = \sum_{j=1}^k \varphi_j^m(z')$. Then

$$S_m(z') = \frac{1}{2\pi i} \int_{|\zeta| = \delta_n} \frac{\zeta^m (\partial f / \partial \zeta)(z', \zeta)\, d\zeta}{f(z', \zeta)},$$

by a formula in Ahlfors, 153–154. It follows that S_m are holomorphic. Since a_j are polynomials in S_m, we see that a_j are holomorphic. Thus p is a Weierstrass polynomial. Finally, u is holomorphic since

$$u(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_n} \frac{(f/p)(z', \zeta)\, d\zeta}{\zeta - z_n}.$$

□

Exercise 1.22. State and prove a Weierstrass preparation theorem for formal power series.

Remark. Weierstrass preparation theorem for C^∞ functions: Malgrange preparation theorem. See J.N. Mather, Stability of C^∞ mappings I, Ann. Math. (2) 87(1968), 89-104, or Hörmander, The analysis of linear partial differential operators I, Springer, Section 7.5.

Theorem 1.23. *(Weierstrass Division Theorem) Let $p(z', z_n)$ be a Weierstrass polynomial of degree k in z_n. Then for each $f \in \mathscr{O}_n$ there exist $h \in \mathscr{O}_n$ and $r \in \mathscr{O}_{n-1}[z_n]$ with $\deg r < k$ such that $f = p \cdot h + r$.*

Proof. Choose a polydisc $\Delta^n(0, \delta)$, where $\delta = (\delta', \delta_n)$, such that p and f are defined in some neighborhood of $\overline{\Delta}^n(0, \delta)$ and

$$p(z', z_n) \neq 0, \text{ if } z' \in \overline{\Delta}^{n-1}(0', \delta') \text{ and } |z_n| = \delta_n.$$

Let

$$h(z) = \frac{1}{2\pi} \int_{|\zeta_n|=\delta_n} \frac{f(z', \zeta_n)\, d\zeta_n}{p(z', \zeta_n)(\zeta_n - z_n)}$$

and $r = f - ph$. Then $h, r \in \mathscr{O}_n$. It follows from

$$r(z', z_n) = \frac{1}{2\pi} \int_{|\zeta_n|=\delta_n} \frac{f(z', \zeta_n)}{p(z', \zeta_n)} \frac{p(z', \zeta_n) - p(z', z_n)}{\zeta_n - z_n}\, d\zeta_n$$

that $r \in \mathscr{O}_{n-1}[z_n]$ with $\deg r < k$. □

A sequence $\{f_j\}$ in $C(D)$ is said to converge compactly to a function f if $\{f_j\}$ converges to f uniformly on each compact subset of D.

Theorem 1.24. *Suppose that* $\{f_j\} \subset \mathscr{O}(D)$ *converges compactly to* f*. Then* $f \in \mathscr{O}(D)$ *and* $\lim D^\alpha f_j = D^\alpha f$*.*

Proof. This is a consequence of Cauchy integral formula for polydiscs. □

A sequence $\{K_j\}$ of compact subsets of D is said to be a *normal exhaustion* of D if $\cup K_j = D$ and if $K_j \subset \operatorname{int} K_{j+1}$ for all $j \geq 1$.

Fix a normal exhaustion $\{K_j\}$ of D. Define a distance function $\lambda(f, g)$ on $C(D)$ by

$$\lambda(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}.$$

Proposition 1.25. *A sequence* $\{f_j\}$ *in* $C(D)$ *converges compactly to* f *if and only if* $\lim \lambda(f_j, f) = 0$.

With the metric λ, $C(D)$ and $\mathscr{O}(D)$ are complete metric spaces. Hence they are Fréchét spaces.

Theorem 1.26. *(Montel Theorem) Let* $\{f_j\} \subset \mathscr{O}(D)$ *be a sequence such that for each compact* $K \subset D$*, the sequence* $\{|f_j|_K\}$ *is bounded. Then* $\{f_j\}$ *has a subsequence that converges uniformly on each compact subset of* D*.*

Proof. The proof is the same as for $n = 1$. The Cauchy estimates tell us that the sequences of partial derivatives are uniformly bounded on fixed K_k. By the mean value theorem, the sequence is uniformly equicontinuous on K_k. By Arzelà-Ascoli theorem, the sequence has a subsequence uniformly convergent on K_k. Then a diagonal sequence argument yields a desired subsequence. □

Exercise 1.27. Let $I = [0, 1]$. Construct a bounded sequence in $C(I)$ that does not have a convergent subsequence.

Exercise 1.28. Let S be a closed subset of a complete metric space. Then S is compact if and only if every sequence in S has a convergent subsequence.

Hint: Each sequence in S contains a convergent subsequence if and only if for each $\varepsilon > 0$, there is a finite ε-net for S. □

A subset S of a topological vector space is said to be bounded if for each neighborhood U of the origin there is an $r > 0$ such that $S \subset rU$.

Exercise 1.29. A subset S of $\mathscr{O}(D)$ is compact if and only if S is closed and bounded.

Hint: When S is bounded, the functions in S are uniformly equicontinuous on each compact subset of D. □

Exercise 1.30. No neighborhood of 0 in $\mathscr{O}(\Delta)$ is bounded.

Hint: Let U be a neighborhood of 0 in $\mathscr{O}(\Delta)$. Show that there is a compact subset K of Δ and an $\varepsilon > 0$ such that $U \supset U_{K,\varepsilon} := \{f \in \mathscr{O}(\Delta) : |f|_K < \varepsilon\}$. Show that $U_{K,\varepsilon}$ is not bounded.

Exercise 1.31. (Hurwitz Theorem) If the functions $\{f_j\}$ are analytic and non-vanishing on a domain $D \subset \mathbb{C}^n$, and if $\{f_j\}$ converges to f, uniformly on every compact subset of D, then f is identically zero, or non-vanishing on D.

2. Holomorphic Mappingss

Definition 2.1. A holomorphic map (or mapping) $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$ is a map of the form $f = (f_1, \dots, f_m)$, where $f_j \in \mathscr{O}(D)$. If $Q \subset \mathbb{C}^m$, a holomorphic map $f : D \to Q$ is a holomorphic map $f : D \to \mathbb{C}^m$ with $f(D) \subset Q$.

Example 2.2. There is no "open image theorem" for holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ when $n > 1$. Consider $f(z, w) = (z, zw)$.

2.3. $\mathbb{R}$-linear transformations on $\mathbb{C}^n$. Recall that we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the following way:

$$(z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n) \approx (x_1, \dots, x_n, y_i, \dots, y_n).$$

Consider a $\mathbb{R}$-linear map $L : \mathbb{C}^n \to \mathbb{C}^m$. It is represented by the matrix

$$L_{\mathbb{R}} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathbb{R}^{2m\times 2n},$$

where $A, B, C, D \in \mathbb{R}^{m\times n}$. The multiplying-by-$i$ operator on $\mathbb{C}^n$, considered as a $\mathbb{R}$-linear operator on $\mathbb{R}^{2n}$, is represented by the matrix

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

So L is $\mathbb{C}$-linear if and only if $J_m L_{\mathbb{R}} = L_{\mathbb{R}} J_n$, or

$$\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} -B & -D \\ A & C \end{pmatrix} = \begin{pmatrix} C & -A \\ D & -B \end{pmatrix},$$

or

$$C = -B, \quad D = A.$$

We have thus obtained that

$$L \text{ is } \mathbb{C}\text{-linear} \quad \Leftrightarrow \quad C = -B, \quad D = A. \tag{1}$$

It follows that L is $\mathbb{C}$-linear if and only if it is represented by a matrix of the form

$$L_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{2}$$

If that is the case, the $\mathbb{C}$-linear map L is represented by the matrix $L_{\mathbb{C}} = (A + iB) \in \mathbb{C}^{m\times n}$. When $m = n$, we obtain

$$|\det L_{\mathbb{C}}|^2 = \det L_{\mathbb{R}}$$

from the identity

$$\begin{pmatrix} I & iI \\ 0 & I \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} I & -iI \\ 0 & I \end{pmatrix} = \begin{pmatrix} A + iB & 0 \\ B & A - iB \end{pmatrix}.$$

2.4. Differential of a C^1 map between complex affine spaces.

Consider a C^1 map $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$. The differential $df(a)$ of f at $a \in D$ is the unique $\mathbb{R}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ which approximates f near a in the sense that

$$f(z) = f(a) + df(a)(z-a) + o(|z-a|).$$

The $\mathbb{R}$-linear map $df(a)$ is represented by the $2m \times 2n$ matrix

$$df(a)_{\mathbb{R}} = \begin{pmatrix} U_X(a) & U_Y(a) \\ V_X(a) & V_Y(a) \end{pmatrix},$$

where $f_j = u_j + iv_j$, and

$$U_X(a) = \frac{\partial(u_1,\dots,u_m)}{\partial(x_1,\dots,x_n)} = (\frac{\partial u_j}{\partial x_k})_{j=1,\dots,m;\ k=1,\dots,n}, \quad \text{etc.}$$

By (1), we have

$$df(a) \text{ is } \mathbb{C}\text{-linear} \quad \Leftrightarrow \quad U_X(a) = V_Y(a), \ \ U_Y(a) = -V_X(a).$$

I.e., $df(a)$ is $\mathbb{C}$-linear if and only if the Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k}(a) = \frac{\partial v_j}{\partial y_k}(a), \ \ \frac{\partial u_j}{\partial y_k}(a) = -\frac{\partial v_j}{\partial x_k}(a), \ \ j = 1,\dots,m; \ k = 1,\dots,n$$

are satisfied. We have proved the following theorem.

Theorem 2.5. *A C^1 map $D \to \mathbb{C}^m$ is holomorphic on D if and only if the differential $df(a)$ is $\mathbb{C}$-linear at every point $a \in D$. If that is the case then the $\mathbb{C}$-linear map $df(a)$ is represented by the complex Jacobian matrix*

$$df(a)_{\mathbb{C}} = \frac{\partial(u_1,\dots,u_m)}{\partial(x_1,\dots,x_n)}(a) + i\frac{\partial(v_1,\dots,v_m)}{\partial(x_1,\dots,x_n)}(a) = \frac{\partial(f_1,\dots,f_m)}{\partial(z_1,\dots,z_n)}(a).$$

If f is holomorphic and if $m = n$ then

$$|\det df(a)_{\mathbb{C}}|^2 = \det df(a)_{\mathbb{R}}. \quad \Box \tag{3}$$

Calculation of $\partial f_1/\partial z_1$:

$$\frac{\partial f_1}{\partial z_1} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1})(u_1 + iv_1) = \frac{1}{2}(\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1}) + \frac{i}{2}(\frac{\partial v_1}{\partial x_1} - \frac{\partial u_1}{\partial y_1}).$$

The right side equals

$$\frac{\partial u_1}{\partial x_1} + i\frac{\partial v_1}{\partial x_1}$$

since the Cauchy-Riemann equations are satisfied. Also, $(\partial f_1/\partial z_1)(a) = h'(a_1)$, where h is the holomorphic function defined by $h(w) = f_1(w, a_2, \dots, a_n)$.

2.6. Split of the differential $df(a)$.

Consider a $\mathbb{R}$-linear map $L : \mathbb{C}^n \to \mathbb{C}^m$. The map L is said to be conjugate $\mathbb{C}$-linear if $L(iz) = -iL(z)$ for $z \in \mathbb{C}^n$. Define $\mathbb{R}$-linear maps $P(L), Q(L) : \mathbb{C}^n \to \mathbb{C}^m$ by

$$P(L)(z) = \frac{1}{2}(L(z) - iL(iz)), \quad Q(L)(z) = \frac{1}{2}(L(z) + iL(iz)).$$

Then we have

(i) $L = P(L) + Q(L)$,

(ii) $P(L)$ is $\mathbb{C}$-linear, $Q(L)$ is conjugate $\mathbb{C}$-linear,
(iii) L is $\mathbb{C}$-linear if and only if $Q(L) = 0$,
(iv) L is conjugate $\mathbb{C}$-linear if and only if $P(L) = 0$.

Consider a C^1 map $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$. The differential $df(a)$ is a $\mathbb{R}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$. We have

$$df(a) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)\, dx_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(a)\, dy_j.$$

Interpretation: $df(a)$ is represented by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) & \frac{\partial f_1}{\partial y_1}(a) & \cdots & \frac{\partial f_1}{\partial y_n}(a) \\ \cdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) & \frac{\partial f_m}{\partial y_1}(a) & \cdots & \frac{\partial f_m}{\partial y_n}(a) \end{pmatrix} \in \mathbb{C}^{m\times 2n}.$$

The $\mathbb{R}$-linear transformation $(\partial f/\partial x_j)(a) : \mathbb{R}^1 \to \mathbb{C}^m$ is represented by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_j}(a) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(a) \end{pmatrix} \in \mathbb{C}^{m\times 1},$$

which naturally extends to a $\mathbb{C}$-linear transformation $\mathbb{C}^1 \to \mathbb{C}^m$. Similar for $(\partial f/\partial y_j)(a)$. The differential dx_j (respectively, dy_j) is the linear transformation $\mathbb{C}^n \to \mathbb{R}^1$ represented by

$$(0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{1\times 2n},$$

where 1 is at the j-th (respectively, $(n+j)$-th) position. Of course dx_j, dy_j can be considered $\mathbb{R}$-linear maps $\mathbb{C}^n \to \mathbb{C}^1$. Let

$$dz_j = dx_j + i dy_j, \quad d\overline{z}_j = dx_j - i dy_j.$$

We now calculate $P(df(a))$ and $Q(df(a))$. We have

$$\begin{aligned}
dx_j(z) &= \Re z_j \\
dx_j(iz) &= -\Im z_j = -dy_j(z) \\
P(dx_j)(z) &= \frac{1}{2}(dx_j(z) - i dx_j(iz)) \\
&= \frac{1}{2}(dx_j(z) + i dy_j(z)) \\
P(dx_j) &= \frac{1}{2}(dx_j + i dy_j) = \frac{1}{2} dz_j \\
P(dy_j) &= \frac{1}{2}(dy_j - i dx_j) = -\frac{i}{2} dz_j \\
P(df(a)) &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) P(dx_j) + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(a) P(dy_j) \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) \cdot \frac{1}{2} dz_j - \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(a) \cdot \frac{i}{2} dz_j \\
P(df(a)) &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)\, dz_j \\
Q(df(a)) &= \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}(a)\, d\overline{z}_j.
\end{aligned}$$

Let

$$\partial f(a) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)\, dz_j, \quad \overline{\partial} f(a) = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}(a)\, d\overline{z}_j.$$

Then we have the following theorem.

Theorem 2.7. *Let $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$ be a C^1 map and let $a \in D$. Then*
(i) $df(a) = \partial f(a) + \overline{\partial} f(a)$.
(ii) $\partial f(a)$ is $\mathbb{C}$-linear, $\overline{\partial} f(a)$ is conjugate $\mathbb{C}$-linear,
(iii) $df(a)$ is $\mathbb{C}$-linear if and only if $\overline{\partial} f(a) = 0$,
(iv) $df(a)$ is conjugate $\mathbb{C}$-linear if and only if $\partial f(a) = 0$. □

Exercise 2.8.

$$\det df(a)_{\mathbb{R}} = \det \begin{pmatrix} F_Z & F_{\overline{Z}} \\ \overline{F}_Z & \overline{F}_{\overline{Z}} \end{pmatrix}$$

Theorem 2.9. *Let $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$ and $g : \Omega \subset \mathbb{C}^m \to \mathbb{C}^\ell$ be holomorphic maps with $f(D) \subset \Omega$. Then $g \circ f$ is a holomorphic map and*

$$d(g \circ f)(a) = dg(f(a)) \cdot df(a) \tag{4}$$

for each $a \in D$.

Proof. By the composition theorem for C^∞ maps, $g \circ f$ is C^∞ and (4) holds. Since $dg(f(a))$ and $df(a)$ are $\mathbb{C}$-linear their composition is $\mathbb{C}$-linear. Therefore $g \circ f$ is holomorphic. □

Corollary 2.10. *Let $f \in \mathscr{O}(D)$, where D is a domain in $\mathbb{C}^n$, and let L be a linear coordinate change of $\mathbb{C}^n$ ($L(z) = Az + b$ with $A \in GL(n, \mathbb{C})$ and $b \in \mathbb{C}^n$). Then $f \circ L \in \mathscr{O}(L^{-1}(D))$.*

Exercise 2.11. Let D be a domain in $\mathbb{C}^n$ and let $\tilde{D} = \{\overline{z} : z \in D\}$. Suppose that $f \in \mathscr{O}(D \times \tilde{D})$ and $f(z, \overline{z}) = 0$ for each $z \in D$. Prove that $f \equiv 0$.

Theorem 2.12. *(Uniqueness Theorem) Let $\Omega \subset \mathbb{C}^n$ be a domain and let $f \in \mathscr{O}(Om)$. Suppose that there is a point $a \in \Omega$ where $D^\alpha f(a) = 0$ for all $\alpha \in \mathbb{N}^n$. Then $f \equiv 0$.*

Proof. Suppose that $f \not\equiv 0$. Then for each neighborhood U of a there is a $w \in U$ with $f(w) \neq 0$. After a coordinate change we may assume that f is z_n- regular at a. By Weierstrass preparation theorem, there is a positive integer k such that $\partial^k f/\partial z_n^k(a) \neq 0$, contradicting the hypothesis. Therefore, $f \equiv 0$. □

Theorem 2.13. *(Riemann Extension Theorem) Let D be a domain in $\mathbb{C}^n$ and let $f \in \mathscr{O}(D)$ be a holomorphic function not identically 0. Let $E = \{z \in D : f(z) = 0\}$. Suppose that $g \in \mathscr{O}(D \setminus E)$ is locally bounded near E in the sense that for each $z \in E$ there is a neighborhood U of z such that g is bounded on $U \setminus E$. Then g extends to a holomorphic function on D.*

Proof. It suffices to prove that for every $z \in E$ there is a neighborhood U of z such that g extends to a holomorphic function on U. Assume that $0 \in E$. By Corollary 2.10, we may assume that f is z_n-regular at 0 (i.e. $f(0', \cdot)$ is not identically 0). We will prove that g extends holomorphically to a neighborhood of 0. Choose $\delta, \eta > 0$ so that $Q := \{(z', z_n) : |z'| \leq \eta, |z_n| \leq \delta\} \subset D$, g is bounded on $Q \setminus E$, and f is non-vanishing on the set

$$\{(0', z_n) : 0 < |z_n| \leq \delta\} \cup \{(z', z_n) : |z'| \leq \eta, |z_n| = \delta\}.$$

For each fixed z' the function $g(z', \cdot)$ extends to a holomorphic function on $\Delta(0, \delta)$. The function

$$h(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{g(z', \zeta)\, d\zeta}{\zeta - z_n}$$

is holomorphic on $V := \{(z', z_n) : |z'| < \eta, |z_n| < \delta\}$ and coincides with g on $V \setminus E$. □

Corollary 2.14. *The set $D \setminus E$ is connected.*

Proof. Suppose that $D \setminus E = U_1 \cup U_2$, where U_1 and U_2 are nonempty open sets with $U_1 \cap U_2 = \emptyset$. Define a function h on $D \setminus E$ by $g(z) = 1$ for $z \in U_1$, $g(z) = 2$ for $z \in U_2$. Then $g \in \mathscr{O}(D \setminus E)$. By the theorem, g extends to a function in $\mathscr{O}(D)$, contradicting the identity theorem. □

Theorem 2.15. *The ring $\mathscr{O}_n$ is a UFD.*

Proof. We proceed by induction on n. When $n = 1$, $\mathscr{O}_1$ obviously is a UFD. Assume that $n > 1$ and $\mathscr{O}_{n-1}$ is a UFD. Let $f \in \mathscr{O}_n$. We assume that f is z_n-regular, i.e., $f(0', z_n) \not\equiv 0$. By Weierstrass preparation theorem, $f = pu$, where u is a unit in $\mathscr{O}_n$ and $p \in \mathscr{O}_{n-1}[z_n]$ is a Weierstrass polynomial. The polynomial ring $\mathscr{O}_{n-1}[z_n]$ is a UFD by the Gauss lemma, so we can write p as a product of irreducible elements $q_1, \ldots, q_m \in \mathscr{O}_{n-1}[z_n]$, where $q_1, \ldots, q_m$ are uniquely determined up to multiplication by units. Since p is monic, we may assume that each q_j is monic. Since p is a a Weierstrass polynomial, all q_j have to be Weierstrass polynomials. We claim that q_j are irreducible in $\mathscr{O}_n$. Suppose that $q_1 = g_1 g_2$, where g_1, g_2

are non-units in $\mathscr{O}_n$. Then $g_1 = s_1 v_1$, $g_2 = s_2 v_2$ by Weierstrass preparation theorem. It follows that $q_1 = s_1 s_2$ and q_1 cannot be irreducible in $\mathscr{O}_{n-1}[z_n]$. Therefore, q_j are irreducible in $\mathscr{O}_n$.

Now suppose that f is the product of irreducible elements $f_1, \ldots, f_k$ in $\mathscr{O}_n$. Each f_j must be z_n-regular. By Weierstrass preparation theorem, $f_j = p_j u_j$, so $f = pu = \Pi p_j \cdot \Pi u_j$. By the uniqueness, $p = \Pi p_j$. Since $\mathscr{O}_{n-1}[z_n]$ is a UFD, it follows that the p_j are the same, up to units and up to the order, as the q_j. Thus f ha a unique factorization in $\mathscr{O}_n$. □

Theorem 2.16. *(Implicit Mapping Theorem) $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$ holomorphic. $m \leq n$, $f(0) = 0$.*

$$\det \left(\frac{\partial f_j}{\partial z_k}(0) \right)_{\substack{j=1,\ldots,m \\ k=n-m+1,\ldots,n}} \neq 0. \tag{5}$$

Then there are $\varepsilon, \eta > 0$, and a holomorphic map $h : B_{n-m}(0,\varepsilon) \to B_m(0,\eta)$ with the property that for $(z,w) \in B_{n-m}(0,\varepsilon) \times B_m(0,\eta)$, $f(z,w) = 0$ if and only if $w = h(z)$.

Proof. By the C^∞ version of the theorem, there exist $\varepsilon, \eta > 0$ and a C^∞ map h such that the above holds and such that (5) holds with 0 replaced by $(a,b) \in B_{n-m}(0,\varepsilon) \times B_m(0,\eta)$. Since $f(z,h(z)) = 0$, by the chain rule we have

$$df(a,h(a))(v,0) + df(a,h(a))(0,dh(a)(v)) = 0, \quad \text{for } a \in B_{n-m}(0,\varepsilon),\ v \in \mathbb{C}^{n-m}.$$

Since $df(a,h(a))$ is $\mathbb{C}$-linear and since the transformation $w \mapsto df(a,h(a))(0,w)$ is bijective we see that $dh(a)$ is $\mathbb{C}$-linear for $a \in B_{n-m}(0,\varepsilon)$ and h is holomorphic. □

Definition 2.17. A subset M of $\mathbb{C}^n$ is said to be a complex submanifold of $\mathbb{C}^n$ if for each point $a \in M$ there is an integer r, with $0 \leq r \leq n$, a neighborhood U of a in $\mathbb{C}^n$, and a holomorphic map $f : U \to \mathbb{C}^m$, where $m = n - r$, such that

$$\det \left(\frac{\partial f_j}{\partial z_k}(a) \right)_{\substack{j=1,\ldots,m \\ k=r+1,\ldots,n}} \neq 0 \tag{6}$$

and

$$M \cap U = \{z \in U : f(z) = 0\}.$$

The number r is called the dimension of M at a.

Theorem 2.18. *(Inverse Mapping Theorem) Let $g : D \subset \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map with $0 \in D$ and $\det dg(0)_{\mathbb{C}} \neq 0$. Then there is an $\varepsilon > 0$ with $B_n(0,\varepsilon) \subset D$ and a holomorphic map $h : g(B_n(0,\varepsilon)) \to \mathbb{C}^n$ such that $h \circ g(z) = z$ for $z \in B_n(0,\varepsilon)$ and $g \circ h(w) = w$ for $w \in g(B_n(0,\varepsilon))$.*

Proof. Consider the map $f(z,w) = -z + g(w)$ and apply the implicit mapping theorem. □

Exercise 2.19. Use the inverse mapping theorem to prove the implicit mapping theorem.

Hint: Consider the map $g(z) = (z_1, \ldots, z_{n-m}, f(z))$ and apply the inverse mapping theorem. □

Let r be an integer with $0 < r \leq n$ and let $U \subset \mathbb{C}^r$ be an open set. A holomorphic mapping $h : U \to \mathbb{C}^n$ is said to be *non-singular* if h is injective and the $\mathbb{C}$-linear transformation $dh(a)$ has complex rank r for every $a \in U$.

Theorem 2.20. *(Parametrization Theorem) A subset M of $\mathbb{C}^n$ is a complex submanifold if and only if for each point $a \in M$ there are a neighborhood U of a in $\mathbb{C}^n$, an open ball $B_r(0,\varepsilon)$, and a nonsingular holomorphic map $h : B_r(0,\varepsilon) \to \mathbb{C}^n$ such that $h(B_r(0,\varepsilon)) = M \cap U$.*

Proof. Suppose that there is a nonsingular holomorphic map $h : B_r(0,\varepsilon) \to \mathbb{C}^n$ such that $h(B_r(0,\varepsilon)) = M \cap U$ and $h(0) = a$. Let $h = (f_1,\dots,f_n)$. Without loss of generality, we assume that $(f_1,\dots,f_r)$ has a nonzero Jacobian at 0. By composing with the inverse mapping of $(f_1 - a_1,\dots,f_r - a_r)$, we assume that $f_1 = z_1 + a_1,\dots,f_r = z_r + a_r$. So locally M is defined by $z_j - f_j(z_1 - a_1,\dots,z_r - a_r) = 0, j = r+1,\dots,n$.

Conversely, suppose that $f = (f_1,\dots,f_m)$ is the defining map of M at a, so (6) holds. By the implicit mapping theorem, there are holomorphic mappings $g_j(z_1,\dots,z_r)$, $j = 1,\dots,m$, defined near $(a_1,\dots,a_r)$ such that $f = 0$ if and only if $z_{r+j} = g_j$, $j = 1,\dots,m$. The map $h(z_1,\dots,z_r) = (z_1,\dots,z_r,g_1,\dots,g_m)$ is the desired parametrization. □

Definition 2.21. A holomorphic map $f : D \subset \mathbb{C}^n \to \mathbb{C}^n$ is said to be *biholomorphic* if f is injective and f^{-1} is holomorphic on $f(D)$. Domains D and Ω are said to be *biholomorphically equivalent* if there is a biholomorphic map $f : D \to \mathbb{C}^n$ with $f(D) = \Omega$. An *automorphism* of a domain D is a biholomorphic map from D onto D.

Theorem 2.22. *(Rank Theorem) Let $f : D \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic map with $0 \in D$, $f(0) = 0$. Suppose that the rank of the complex Jacobian matrix $df(a)_{\mathbb{C}}$ is constant r for each $a \in D$. Then there is a biholomorphic map g defined in some neighborhood of 0 in $\mathbb{C}^n$ and a biholomorphic map h defined in some neighborhood of 0 in $\mathbb{C}^m$ such that $h \circ f \circ g(z_1,\dots,z_n) = (z_1,\dots,z_r,0,\dots,0)$ for all $(z_1,\dots,z_n)$ in some neighborhood of 0.*

Proof. Without loss of generality we assume that

$$\det\left(\frac{\partial f_j}{\partial z_k}(0)\right)_{\substack{j=1,\dots,r\\k=1,\dots,r}} \neq 0.$$

The holomorphic map $G(z_1,\dots,z_n) = (f_1(z),\dots,f_r(z),z_{r+1},\dots,z_n)$ has a nonsingular Jacobian matrix at 0. By the inverse mapping theorem, G has a holomorphic inverse g in some neighborhood of 0. Now $f \circ g(z_1,\dots,z_n) = (z_1,\dots,z_r,q_{r+1}(z),\dots,q_m(z))$. Since $d(f \circ g)$ has constant rank r, we see that $\partial q_j/\partial z_k \equiv 0$ for $k > r$. So q_j depend on only $z_1,\dots,z_r$. Let $h(w_1,\dots,w_m) = (w_1,\dots,w_r,w_{r+1} - q_{r+1}(w_1,\dots,w_r),\dots,w_m - q_m(w_1,\dots,w_r))$. □

Exercise 2.23. Derive Implicit Mapping Theorem from Rank Theorem.

Exercise 2.24. $\{w^2 = z^3\}$ is not a submanifold.

Exercise 2.25. If $f \in \mathscr{O}_n$ is irreducible, $n > 1$, $df(0) = 0$, then $\{f = 0\}$ is not a submanfold.

2.26. Digression: Resultant and Discriminant.

If $p, q \in K[t]$, then

$$R(p,q) = a_\ell^m b_m^\ell \prod_{p(x)=0} \prod_{q(y)=0} (y - x),$$

where $p(t) = a_\ell t^\ell + \cdots$, $q(t) = b_m t^m + \cdots$. The roots are in the algebraic closure $\overline{K}$ of K. It is clear that $R(p,q) = 0$ if and only if p, q have a common root or common factor. We can see that

$$R(p,q) = b_m^\ell \prod_{q(y)=0} p(y).$$

Let $S = \det(C \quad D)$, $C = (c_{jk}) \in K^{(\ell+m)\times m}$, $D = (d_{jk}) \in K^{(\ell+m)\times \ell}$, c_{jk} is the coefficient of t^{j-1} in $t^{k-1}p(t)$, d_{jk} is the coefficient of t^{j-1} in $t^{\ell-1}q(t)$.

If p, q have a common root α, then $(1, \alpha, \dots, \alpha^{\ell+m-1})(C \quad D) = 0$, and hence $S = 0$. Thus $S = R(p, q)Q$, where Q is a polynomial in a_μ and b_ν. The highest power of a_0 in R is $a_0^m b_m^\ell$, and the same is true for S. It follows that $R = S$.

The discriminant of p is $R(p, p')$. It is nonzero if and only if p has no repeated irreducible factors.

Theorem 2.27. *Let $n > 1$, let $f \in \mathscr{O}_n$ be z_n-regular, and let $Z = \{f = 0\}$. Then there is an integer $k > 0$, and a non-zero $g \in \mathscr{O}_{n-1}$ defined in some neighborhood V of 0 in $\mathbb{C}^{n-1}$ such that for each $a' \in V \setminus Q$, where $Q := \{g = 0\}$, $f(z', \cdot)$ has k distinct roots. Moreover, there is a neighborhood W of a' in $\mathbb{C}^{n-1}$ and k holomorphic functions $h_j \in \mathscr{O}(W)$ such that for each $z' \in W$, $f(z', h_j(z')) = 0$ for $j = 1, \dots, k$.*

Proof. Without loss of generality, we assume that f is a Weierstrass polynomial p and it is the product of distinct irreducibles. Let $k = \deg p$. Choose a neighborhood V of $0'$ in $\mathbb{C}^{n-1}$ and $\eta > 0$ such that f, p, u are defined on $V \times \Delta(0, \eta)$, and for each $z' \in V$, $p(z', \cdot)$ has exactly k roots in $\Delta(0, \eta)$ counting multiplicities. Since p and $\partial p/\partial z_n$ have no common factors in $\mathscr{O}_{n-1}[z_n]$, their resultant g (which is the discriminant of p) is a nonzero element of $\mathscr{O}_{n-1}$ defined in V, and $Q := \{z' \in V : g(z') = 0\}$ is nowhere dense in V. The expression of the resultant as a determinant of coefficients tells us that if g is the resultant of $p(z', z_n)$ and $(\partial p/\partial z_n)(z', z_n)$ in $\mathscr{O}_{n-1}[z_n]$ then, for fixed z', $g(z')$ is the resultant of $p(z', \cdot)$ and $(\partial p/\partial z_n)(z', \cdot)$ in $\mathbb{C}[z_n]$. This means that $p(a', \cdot)$ has k distinct roots if and only if $a' \in V \setminus Q$. The last statement is obtained by applying the Weierstrass preparation theorem to p at each of the roots of $p(a', \cdot)$. □

Theorem 2.28. *Let $f : D \subset \mathbb{C}^n \to \mathbb{C}^n$ be an injective holomorphic mapping. Then* $\det df$ *is nonvanishing on D.*

Proof. When $n = 1$ the statement is true: if $f'(a) = 0$, then $f(x) = f(a) + g(x - a)^k$ with $k > 1$, f cannot be injective.

Let $n > 1$. Suppose that $df(a) \neq 0$. Then $(\partial f_n/\partial z_n)(a) \neq 0$ after a coordinate change. We may assume that $f_n(z) = z_n$ if we use $z_1, \dots, z_{n-1}, f_n(z)$ as the coordinates. Since the map $z' \mapsto f(z', a_n)$ is injective, its differential is nonsingular by the induction hypothesis. Thus $\det df(a) \neq 0$. In summary, if $df(a) \neq 0$ then $\det df(a) \neq 0$.

Suppose that $Z := \{\det df = 0\}$ is non-empty. Then $df = 0$ on Z. By Theorem 2.27, there is an injective holomorphic mapping $h : B_{n-1} \to \mathbb{C}^n$ with $h(B_{n-1}) \subset Z$. Since $d(f \circ h) \equiv 0$, $f \circ h$ is constant, contradicting the hypothesis that f is injective. Therefore Z is empty. □

Definition 2.29. Let D be a domain in $\mathbb{C}^n$. The Carathéodory and Kobayashi infinitesimal pseudo-metrics are functions from $D \times \mathbb{C}^n$ to $[0, \infty)$ defined by

$$C_D(z, v) = \sup\{|dg(z)(v)| : g \in \mathscr{O}(D, \Delta), g(z) = 0\},$$
$$K_D(z, v) = \inf\{|u| : u \in \mathbb{C}, f \in \mathscr{O}(\Delta, D), f(0) = z, df(0)(u) = v\}.$$

The Kobayashi *indicatrix* of D at z is

$$I_{D,z} := \{v \in \mathbb{C}^n : K_D(z, v) < 1\}.$$

Extremal maps exist when D is bounded.

It was proved by Royden (H.L. Royden, Remarks on the Kobayshi metric, pp. 125–137, Lecture notes in Math., v. 185, 1971) that every Kobayashi hyperbolic complex manifold is infinitesimally Kobayashi non-degenerate. The converse is false (M. Jarnicki, P. Pflug, Invariant distances and metrics in complex analysis, Walter de Gruyter & Co., Berlin, 1993, Remark 3.5.11).

Theorem 2.30. $C_D \le K_D$.

Proof. Let $z \in D$ and $v \in \mathbb{C}^n$. Let $\varepsilon > 0$ be given. There is an $f \in \mathscr{O}(D, \Delta)$ and $u \in \mathbb{C}^n$ such that $f(0) = z$, $df(0)(u) = v$, and $|u| < K_D(z, v) + \varepsilon$. There is a $g \in \mathscr{O}(D, \Delta)$ such that $g(z) = 0$, and $|dg(z)(v)| > C_D(z, v) - \varepsilon$. By Schwarz lemma, $|d(g \circ f)(u)| \le |u|$. It follows that

$$C_D(z, v) - \varepsilon < |dg(z)(v)| = |d(g \circ f)(u)| \le |u| < K_D(z, v) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields that $C_D(z, v) \le K_D(z, v)$. □

Theorem 2.31. *The Kobayashi infinitesimal pseudo-metric is decreasing under holomorphic maps in the sense that if $f \in \mathscr{O}(D, \Omega)$ then $K_\Omega(f(z), df(z)(v)) \le K_D(z, v)$.*

Proof. Let $\varepsilon > 0$. Choose $h \in \mathscr{O}(\Delta, D)$ and $u \in \mathbb{C}^n$ so that $h(0) = z$, $dh(0)(u) = v$, and $K_D(z, v) + \varepsilon > |u|$. Since $f \circ h \in \mathscr{O}(\Delta, \Omega)$, and $d(f \circ h)(0)(u) = df(z)(v)$, we see that $|u| \ge K_\Omega(f(z), df(z)(v))$. Thus $K_D(z, v) + \varepsilon > K_\Omega(f(z), df(z)(v))$ for each $\varepsilon > 0$. □

Corollary 2.32. *The Kobayashi metric is invariant under holomorphic maps in the sense that if f is a biholomorphic mapping from D onto Ω then $K_\Omega(f(z), df(z)(v)) = K_D(z, v)$. Moreover $df(z)(I_{D,z}) = I_{\Omega,f(z)}$.*

Exercise 2.33. Show that $I_{\Delta^n,0} = \Delta^n$.

Exercise 2.34. Show that $I_{B_n,0} = B_n$.

Definition 2.35. An *automorphism* of a domain Ω is a biholomorphic mapping from Ω onto Ω. The set of automorphisms of Ω is denoted $\mathrm{Aut}(\Omega)$.

The set $\mathrm{Aut}(\Omega)$ is a group. When Ω is a bounded domain, $\mathrm{Aut}(\Omega)$ is a Lie group.

Exercise 2.36. Show that $Aut(\Delta^n)$ acts on Δ^n transitively.

Theorem 2.37. *Let $n > 1$. Then Δ^n and B_n are not holomorphically equivalent.*

Proof. Suppose that g is a biholomorphic mapping from B_n onto Δ^n. Let $a = g(0)$. By Exercise 2.36, there is an $h \in \mathrm{Aut}(\Delta^n)$ with $h(a) = 0$. Let $f = h \circ g$. Then $f(0) = 0$. By Corollary 2.32, $df(0)(B_n) = \Delta^n$. That is impossible because B_n has a smooth boundary and Δ^n does not. □

Exercise 2.38. Show that $K_{D\times\Omega}((z, w), (u, v)) = \max(K_D(z, u), K_\Omega(w, v))$ and $I_{D\times\Omega,(z,w)} = I_{D,z} \times I_{\Omega,w}$.

A domain $\Omega \subset \mathbb{C}^n$ is said to be a *complete circular domain* if whenever $w \in \Omega$ and $\zeta \in \overline{\Delta}$, we have $\zeta w \in \Omega$.

Exercise 2.39. Let D be a bounded convex complete circular domain in $\mathbb{C}^n$. Then $I_{D,0} = D$.

Hint: For each $a \in \mathbb{C}^n$, $\mathbb{C}a \cap D$ is a holomorphic retract of D.

Remark. T.J. Barth showed that the indicatrix of the Kobayashi infinitesimal metric at the center of a pseudoconvex complete circular domain coincides with this domain. (Proc. AMS, 88(1983), 527–530.) □

Exercise 2.40. Let $0 < \lambda < 1$ and let $f : B_n \to \mathbb{C}^n$ be defined by

$$\begin{aligned} f_1(z) &= \frac{z_1 + \lambda}{1 + \lambda z_1}, \\ f_j(z) &= \frac{\sqrt{1-\lambda^2}}{1 + \lambda z_1} z_j, \quad j = 2, \dots, n. \end{aligned} \tag{7}$$

Show that $f \in \mathrm{Aut}(B_n)$.

3. Power Series

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. For $\alpha \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$ let $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. An expression of the form $g(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$, where $c_\alpha \in \mathbb{C}$, is called a power series in variables $z = (z_1, \dots, z_n)$. The set of such series is denoted by $\mathbb{C}[[z]] = \mathbb{C}[[z_1, \dots, z_n]]$.

Definition 3.1. A series $g \in \mathbb{C}[[z_1, \dots, z_n]]$ is said to *converge* (absolutely) at $a \in \mathbb{C}^n$ if $\sum |c^\alpha a^\alpha| < \infty$ (or the function $\tilde{g}_a : \mathbb{N}^n \to \mathbb{C}$, $\tilde{g}_a(\alpha) = c_\alpha a^\alpha$ is integrable with respect to the counting measure on $\mathbb{N}^n$). In this case, the limit $\lim_{k\to\infty} \sum_{|\alpha|\le k} c_\alpha a^\alpha$ exists; its value is called the *sum of* g *at* a and denoted by $g(a) = \sum_\alpha c_\alpha a^\alpha$.

Remark. Since the index set $\mathbb{N}^n$ does not have a canonical order, we do not discuss conditional convergence of a power series.

Definition 3.2. A series $g(z) = \sum c_\alpha z^\alpha$ is said to *converge uniformly* on $\Gamma \subset \mathbb{C}^n$ if for each $\varepsilon > 0$ there is a finite set $S \subset \mathbb{N}^n$ such that $\sum_{\alpha \notin S} |c_\alpha a^\alpha| < \varepsilon$ for each $a \in \Gamma$.

Proposition 3.3. *Let $g = \sum c_\alpha z^\alpha \in \mathbb{C}[[z_1, \dots, z_n]]$, let $Q \subset \mathbb{C}^n$, and let $\{M_\alpha\}_{\alpha\in\mathbb{N}^n}$ be a family of positive numbers. Suppose that $\sum M_\alpha < \infty$ and that $|c_\alpha z^\alpha| \le M_\alpha$ for $\alpha \in \mathbb{N}^n$ and $z \in Q$. Then g converges uniformly on Q.*

Proof. Exercise. □

Let $\tau : \mathbb{C}^n \to \mathbb{R}^n$ be defined $\tau(z) = (|z_1|, \dots, |z_n|)$. We say $\tau(z) > 0$ if $|z_j| > 0$ for $j = 1, \dots, n$. For $z \in \mathbb{C}^n$ with $\tau(z) > 0$ let $\log \tau(z) = (\log |z_1|, \dots, |z_n|)$.

Theorem 3.4. *Suppose that $\tau(a) > 0$ and the terms of g at a are bounded. Then g converges uniformly on $\overline{\Delta}^n(0, t\tau(a))$ for each $t \in (0, 1)$.*

Proof. By the hypothsis, $|c_\alpha|\tau(a)^\alpha \le M$ for $\alpha \in \mathbb{N}^n$. Fix $0 < t < 1$. For $z \in \overline{\Delta}^n(0, t\tau(a))$,

$$|c_\alpha z^\alpha| \le |c_\alpha|\tau(a)^\alpha t^{|\alpha|} \le M t^{|\alpha|}.$$

Since $\sum M t^{|\alpha|} = M/(1-t)^n$, it follows from Proposition 3.3 that g converges uniformly on $\overline{\Delta}^n(0, t\tau(a))$. □

Definition 3.5. A series $g \in \mathbb{C}[[z_1, \dots, z_n]]$ is said to *converge* if it converges at some point $a \in \mathbb{C}^n$ with $\tau(a) > 0$. The set of series that converge is denoted by $\mathbb{C}\{z\} = \mathbb{C}\{z_1, \dots, z_n\}$.

Exercise 3.6. Let $g \in \mathbb{C}[[z_1, \dots, z_n]]$. Then $g \in \mathbb{C}\{z\}$ if and only if there is a $C > 0$ such that $|c_\alpha| \le C^{|\alpha|+1}$.

Definition 3.7. Let $g \in \mathbb{C}\{z_1, \dots, z_n\}$. The *convergence domain* of g is the the set $\mathscr{Q}(g)$ of the points a such that g converges at each point in some neighborhood of a.

Example 3.8. Let $g(z) = \sum_{j=1}^\infty j^{-2} z^j$ for $z \in \mathbb{C}$. Then $\mathscr{Q}(g) = \Delta$. Note that $1 \notin \mathscr{Q}(\Delta)$ even though g converges at 1.

Definition 3.9. A set $\Omega \subset \mathbb{C}^n$ is said to be *Reinhardt* if whenever $a = (a_1, \dots, a_n) \in \Omega$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, we have $(a_1 e^{it_1}, \dots, a_n e^{it_n}) \in \Omega$.

A domain $\Omega \subset \mathbb{C}^n$ is said to be a *complete Reinhardt domain* if whenever $a = (a_1, \dots, a_n) \in \Omega$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in \overline{\Delta}^n$, we have $(a_1\zeta_1, \dots, a_n\zeta_n) \in \Omega$.

A complete Reinhardt domain Ω is said to be *logarithmically convex* if for $a, b \in \Omega$ with $a, b > 0$ and $0 < \lambda < 1$ we have $a^{1-\lambda} b^\lambda := (a_1^{1-\lambda} b_1^\lambda, \dots, a_n^{1-\lambda} b_n^\lambda) \in \Omega$.

For $s \in \mathbb{R}^n$, we say $s = (s_1, \dots, s_n) \geq 0$ if $s_j \geq 0$ for $j = 1, \dots, n$. Recall that $\tau(z) = (|z_1|, \dots, |z_n|)$; so $\tau(z)^s = |z_1|^{s_1} \cdots |z_n|^{s_n}$. We will take the convention that $0^0 = 1$.

Definition 3.10. Let K be a compact set. The *logarithmically convex hull* of K is defined to be

$$\hat{K}_{\ell c} := \{z \in \mathbb{C}^n : \tau(z)^s \leq \max_{w \in K} \tau(w)^s \quad \text{for all } s \in \mathbb{R}^n \text{ with } s \geq 0\}.$$

Exercise 3.11. Let Ω be a logarithmically convex complete Reinhardt domain and let $b \in \partial\Omega$. Show that there is an $s \in \mathbb{R}^n$ with $s \geq 0$ such that $\tau(b)^s > \tau(z)^s$ for every $z \in \Omega$.

Hint: it takes more reasoning when some of the coordinates of b are 0.

Proposition 3.12. *Let Ω be a complete Reinhardt domain. Then the following are equivalent.*

(i) Ω is logarithmically convex.

(ii) The set $\log \tau(\Omega) := \{\log \tau(z) : z \in \Omega, \tau(z) > 0\} \subset \mathbb{R}^n$ is convex.

(iii) For each compact $K \subset \Omega$, we have $\hat{K}_{\ell c} \subset \Omega$.

Proof. (i) $\Leftrightarrow$ (ii): it is clear.

(iii) $\Rightarrow$ (i): Suppose that $a, b \in \Omega$, $a, b > 0$. Let $0 < \lambda < 1$ and $c = a^{1-\lambda} b^{\lambda}$. We need to show that $c \in \Omega$. Let $K = \{a, b\}$ and $s \in \mathbb{R}^n$ with $s \geq 0$. Then $\tau(c)^s = (\tau(a)^s)^{1-\lambda} (\tau(b)^s)^{\lambda} \leq \max(\tau(a)^s, \tau(b)^s) = \sup_{w \in K} \tau(w)^s$. It follows that $c \in \hat{K}_{\ell c} \subset \Omega$. Therefore Ω is logarithmically convex.

(i) $\Rightarrow$ (iii): Let $a \in \Omega$ with $a > 0$. Suppose that $\hat{K}_{\ell c} \setminus \Omega$ is nonempty, then there is a $c \in \hat{K}_{\ell c} \setminus \Omega$ with $c \geq 0$. Let $\omega(\lambda) = a^{1-\lambda} c^{\lambda}$. Let λ_1 be the least positive number so that $b := \omega(\lambda_1) \in \partial\Omega$. Since Ω is logarithmically convex, there is an $s \in \mathbb{R}$ with $s \geq 0$ such that $b^s > z^s$ for $z \in \Omega$. It follows that $b^s > a^s$. Since b^s is a weighted geometric average of a^s and c^s, it follows that $c^s > b^s > a^s$. Thus $c^s > \max_{w \in K} \tau(w)^s$, contradicting the assumption that $c \in \hat{K}_{\ell c}$. Therefore, $\hat{K}_{\ell c} \subset \Omega$. ☐

Theorem 3.13. *Let $g \in \mathbb{C}\{z\}$. Then $\mathscr{Q}(g)$ is a logarithmically convex complete Reinhardt domain.*

Proof. Let $u \in \mathscr{Q}(g)$ with $\tau(u) > 0$. Then there is an $\varepsilon > 0$ so that g converges at $e^{2\varepsilon} u$. By Theorem 3.4, g converges uniformly on $\overline{\Delta}^n(0, e^{\varepsilon}\tau(u))$. It follows that $\overline{\Delta}^n(0, \tau(u)) \subset \mathscr{Q}(g)$. Thus $\mathscr{Q}(g)$ is a complete Reinhardt domain.

Suppose that $a, b \in \mathscr{Q}(g)$ with $a, b > 0$. Let $0 < \lambda < 1$ and $c = a^{1-\lambda} b^{\lambda}$. Choose $\varepsilon > 0$ so that g converges at $e^{2\varepsilon} a$ and at $e^{2\varepsilon} b$. Thus g converges at $e^{2\varepsilon} c$, because the terms of $g(e^{2\varepsilon} c)$ are weighted geometric means of corresponding terms of $g(e^{2\varepsilon} a)$ and $g(e^{2\varepsilon} b)$. It follows that g converges uniformly on $\overline{\Delta}^n(0, e^{\varepsilon} c)$, which implies that $c \in \mathscr{Q}(g)$. Therefore, $\mathscr{Q}(g)$ is logarithmically convex. ☐

Exercise 3.14. Let $g(z) = \sum a_\alpha z^\alpha \in \mathbb{C}\{z\}$ and let Ω be a domain in $\mathbb{C}^n$. Then $\Omega \subset \mathscr{Q}(g)$ if and only if for each $z \in \Omega$ the set $\{|a_\alpha z^\alpha| : \alpha \in \mathbb{N}^n\}$ is bounded.

Exercise 3.15. Let $g = \sum a_\alpha z^\alpha, h = \sum b_\alpha z^\alpha \in \mathbb{C}\{z\}$. If $|a_\alpha| \leq |b_\alpha|$ for all α, then $\mathscr{Q}(g) \supset \mathscr{Q}(h)$.

Exercise 3.16. Each comvex complete Reinhardt domain is logarithmically convex.

Exercise 3.17. Let $g = \sum z_1^j z_2^k$. Find $\mathscr{Q}(g)$.

Exercise 3.18. Let $g = \sum z_1^k z_2^k$. Find $\mathscr{Q}(g)$.

Exercise 3.19. Let $Q = \{(z,w) \in \mathbb{C}^2 : |z^2 w| < 1, |zw^2| < 1\}$. Find $g \in \mathbb{C}\{z,w\}$ so that $Q = \mathscr{Q}(g)$.

Exercise 3.20. Let $\alpha \in \mathbb{N}^n$. Evaluate $u_\alpha := \max_{z\in\partial B_n} |z^\alpha|$. (Ans. $(\alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n} |\alpha|^{-|\alpha|})^{1/2}$.)

Exercise 3.21. Let $g = \sum u_\alpha^{-1} z^\alpha$. Then $\mathscr{Q}(g) = B_n$.

Exercise 3.22. Suppose that Q is a bounded logarithmically convex complete Reinhardt domain. Let $Q_j = (1-j^{-1})Q$. Let $H = \{r^{(j)}\}$ be the set of rational points $r^{(j)} > 0$ that are in the complement $\tau(\overline{Q})^c$. For each j there is a $c_j > 0$ and $\alpha^{(j)} \in \mathbb{Q}^n$ with $\alpha^{(j)} > 0$ such that $\omega_j(z) := c_j \tau(z)^{\alpha^{(j)}}$ satisfies $\omega_j(r^{(j)}) > 1$ and $\omega_j < 1$ on $\overline{\Omega}_j$. We can choose $\alpha^{(j)}$ successively so that they are pairwise non-proportional. By raising ω_j to suitable powers, we may assume that $\alpha^{(j)} \in \mathbb{N}^n$ and they are all distinct. Let $g = \sum_j c_j z^{\alpha^{(j)}}$. Prove that $Q = \mathscr{Q}(g)$.

Proposition 3.23. *Let $g \in \mathbb{C}\{z\}$. Then $a \in \mathscr{Q}(g)$ if and only if there are $\varepsilon, M > 0$ such that*

$$|c_\alpha|(\tau(a)+\varepsilon)^\alpha \le M, \quad \alpha \in \mathbb{N}^n. \tag{8}$$

Proposition 3.24. *Let $g \in \mathbb{C}\{z\}$ and $\beta \in \mathbb{N}^n$. Let g_β be the series obtained by applying the operator D^β to each term of g. Then $\mathscr{Q}(g_\beta) = \mathscr{Q}(g)$.*

Proof. It suffices to prove the statement for $\beta = (1,0,\dots,0)$. Suppose that $a \in \mathscr{Q}(g)$ and (8) holds. We have

$$\frac{\alpha_1(\tau(a)+\varepsilon/2)^{\alpha-(1,0,\dots,0)}}{(\tau(a)+\varepsilon)^\alpha} \le A := \sup_{\alpha_1\ge 0} \frac{\alpha_1(|a_1|+\varepsilon/2)^{\alpha_1-1}}{(|a_1|+\varepsilon)^{\alpha_1}}.$$

The right side fraction approaches 0 when $\alpha_1 \to \infty$. Thus $A < \infty$. Hence

$$|c_\alpha|\alpha_1(\tau(a)+\varepsilon/2)^{\alpha-(1,0,\dots,0)} \le AM, \quad \alpha \in \mathbb{N}^n.$$

It follows that $a \in \mathscr{Q}(g_\beta)$. Thus $\mathscr{Q}(g_\beta) \supset \mathscr{Q}(g)$. The reversed inclusion is similarly proved. □

Theorem 3.25. *Let $g \in \mathbb{C}\{z\}$ and $\alpha \in \mathbb{N}^n$. Then g, g_α converge, uniformly on compact subsets of $\mathscr{Q}(g)$, to holomorphic functions f, f_α on $\mathscr{Q}(g)$. Moreover, $f_\alpha = D^\alpha f$.*

Proof. By Theorem 3.4, g, g_α converge uniformly on each closed polydisc in $\mathscr{Q}(g)$. Hence they converge uniformly on compact subsets. Since the partial sums are holomorphic and have the derivative relation, we see that f, f_α are holomorphic and $D^\alpha f = f_\alpha$. □

Corollary 3.26. $D^\alpha f(0) = \alpha! c_\alpha$.

Corollary 3.27. *(Identity Theorem) Suppose that $g \in \mathbb{C}\{z\}$ and that g converges to the 0 function in $\mathscr{Q}(g)$. Then all coefficients of g are 0.*

Definition 3.28. Let $f \in \mathscr{O}(\Omega)$ and $a \in \Omega$. The *Taylor series* of f at a is

$$\sum_{\alpha\in\mathbb{N}^n} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha.$$

Theorem 3.29. *Let D be a domain in $\mathbb{C}^n$, let Ω be a logarithmically convex complete Reinhardt domain in $\mathbb{C}^n$, and let $f \in \mathscr{O}(D)$. Suppose that $(a+\Omega) \subset D$. Then the Taylor series of f at a converges to f, uniformly on compact subsets, in $a+\Omega$.*

Proof. Let $\overline{\Delta}^n(0,r) \subset \Omega$. It suffices to show that the Taylor series of f converges to f uniformly on $\Delta^n(a,r)$, since each compact subset of $a+\Omega$ is covered by a finite number of such polydiscs.

Choose $\varepsilon > 0$ so that $\overline{\Delta}(0, e^{\varepsilon} r) \subset \Omega$. Let $M = \max\{|f(z)| : z \in T^n(a, e^{\varepsilon} r)\}$. By Cauchy estimates (Exercise 1.10), for $\alpha \in \mathbb{N}^n$,

$$\frac{|D^\alpha f(a)|}{\alpha!} r^\alpha e^{|\alpha|\varepsilon} \leq M.$$

It follows that

$$\frac{|D^\alpha f(a)|}{\alpha!} |(z-a)^\alpha| \leq M e^{-|\alpha|\varepsilon}, \quad \text{for } z \in \Delta^n(a,r), \quad \alpha \in \mathbb{N}^n.$$

By Proposition 3.3, the Taylor series of f converges to a holomorphic function h uniformly on $\Delta^n(a,r)$. By Corollary 3.26, $D^\alpha h(a) = D^\alpha f(a)$ for $\alpha \in \mathbb{N}^n$. By the uniqueness theorem (Theorem 2.12), $f = h$. Therefore, the Taylor series of f at a converges to f, uniformly on compact subsets, in $a+\Omega$. $\square$

Theorem 3.30. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain containing 0 and let $f \in \mathscr{O}(\Omega)$. Then the Taylor series of f at 0 converges to f, uniformyly on compact subsets, in Ω.*

Proof. Let K be a connected compact set in Ω containing a neighborhood of the origin. Choose $M, \varepsilon > 0$ so that $K \subset B_n(0, M)$ and $(K + B_n(0, 2M\varepsilon)) \subset \Omega$. Let $T_\varepsilon = T^n(0, 1+\varepsilon)$ and

$$g(z) = (2\pi i)^{-n} \int_{T_\varepsilon} \frac{f(t_1 z_1, \ldots, t_n z_n)\, dt_1 \cdots dt_n}{(t_1 - 1) \cdots (t_n - 1)}.$$

Then g is holomorphic in some neighborhood of K. Now we have

$$\frac{1}{(t_1-1)\cdots(t_n-1)} = \sum_\alpha t^{-\alpha-1}$$

uniformly. It follows that $g(z) = \sum f_\alpha(z)$, uniformly on K, where

$$f_\alpha(z) = (2\pi i)^{-n} \int_{T_\varepsilon} f(t_1 z_1, \ldots, t_n z_n) t^{-\alpha-1}\, dt$$

. When z is small, $f_\alpha = z^\alpha D^\alpha f(0)/\alpha!$, hence $g = f$ for small z. The theorem follows from the uniqueness theorem. $\square$

Exercise 3.31. Let $\Omega \subset \mathbb{C}^n$ be a logarithmically -convex complete Reinhardt domain and D a domain in $\mathbb{C}^m$. Let $f \in \mathscr{O}(D \times \Omega)$. Prove that f has a power series expansion

$$f(z,w) = \sum g_\alpha(z) w^\alpha,$$

with $g_\alpha \in \mathscr{O}(D)$, which converges uniformly on compact subsets.

Theorem 3.32. *(H. Cartan (Henri Paul Cartan, 1904–2008), 1931) Let $D \subset \mathbb{C}^n$ be a bounded domain, and let $f \in \mathscr{O}(D, D)$. Suppose that $f(0) = 0$, $f'(0) = I$. Then $f(z) \equiv z$.*

Proof. Suppose not. Then $f(z) = z + f_m(z) + \cdots$, where $f_m(z)$ is the first nonzero homogeneous component with $m > 1$. Then $f^k(z) = z + kf_m(z) + \cdots$. The coefficients of $f_m(z)$ are derivatives of f at 0. Since some subsequence of $\{f^k\}$ converges, the corresponding subsequence of $\{kf_m(z)\}$ converges, which is absurd. □

Theorem 3.33. *(H. Cartan 1931) Let D, Ω be bounded complete circular domains in $\mathbb{C}^n$ and let f be a biholomorphic map from D onto Ω with $f(0) = 0$. Then f is linear.*

Proof. Let $t \in \mathbb{R}$. The differential of the map $z \mapsto f^{-1}(e^{-it}f(e^{it}z))$ at 0 is the identity. So the map is the identity, and $f(e^{it}z) = e^{it}f(z)$. The homogeneous part $f_m(z)$ satisfies $e^{imt}f_m(z) = e^{it}f_m(z)$, hence $f_m(z) = 0$ for $m > 1$. Therefore f is linear. □

Corollary 3.34. *For $n > 1$, Δ^n and B_n are not holomorphically equivalent.*

For $a \in \Omega$, let $\mathrm{Aut}_a := \{h \in \mathrm{Aut}(\Omega) : h(a) = a\}$.

Corollary 3.35. $\mathrm{Aut}_0(B_n) = U(n)$.

Corollary 3.36. $\mathrm{Aut}_0(\Delta^n) = U(1)^n \times S_n$.

For $a \in \Delta^n$, let $\psi_a : \Delta^n \to \mathbb{C}^n$ be defined by

$$\psi_{a,j}(z) = \frac{z_j + a_j}{1 + \overline{a}_j z_j}.$$

Exercise 3.37. Let $f \in Aut(\Delta^n)$. Show that $f = \psi_a \circ L$, where $a = f(0)$ and $L \in Aut_0(\Delta^n)$.

Notation: $\langle z, w\rangle = \sum z_j \overline{w}_j$. For $a \in B_n$, let $s_a = \sqrt{1 - |a|^2}$ and

$$\varphi_a(z) = \frac{a + P_a z + s_a Q_a z}{1 + \langle z, a\rangle}, \tag{9}$$

where

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} z, \quad Q_a z = z - P_a z.$$

Exercise 3.38. Let $f \in Aut(B_n)$. Show that $f = \varphi_a \circ L = M \circ \varphi_{-b}$, where $a = f(0)$, $b = f^{-1}(0)$, and $L, M \in Aut_0(B_n)$.

3.39. Digression: the group $U(n, 1)$. The group $U(n, 1)$ is the group of linear transformations under which the form $X_1\overline{X}_1 + \cdots + X_n\overline{X}_n - X_0\overline{X}_0$ is invariant. So

$$U(n, 1) = \{W \in GL(n + 1, \mathbb{C}) : W^T \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \overline{W} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

In homogeneous coordinates, B_n is given by

$$X_1\overline{X}_1 + \cdots + X_n\overline{X}_n - X_0\overline{X}_0 < 0.$$

Each $W = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ in $U(n, 1)$ gives a map $g_W \in Aut(B_n)$, where

$$g_W(z) = \frac{Az + b}{cz + d}.$$

The map g_W is the identity map if and only if $W \in S^1 := \{e^{it}I : t \in \mathbb{R}\}$. Thus $PU(n, 1) := U(n, 1)/S^1$ is considered a subgroup of $Aut(B_n)$. Since $(n + 1)^2 - 1 = \dim PU(n, 1) \leq$

$\dim Aut(B_n) \le 2n + \dim Aut_0(B_n) = 2n + \dim U(n) = 2n + n^2$, we see that $Aut(B_n) = PU(n,1)$.

If $b = (t, 0, \ldots, 0)$, $t > 0$, $c = b^T$, then $A = \begin{pmatrix} \alpha\sqrt{1+t^2} & 0 \\ 0 & V \end{pmatrix}$, $d = \overline{\alpha}\sqrt{1+t^2}$, where $V \in U(n-1)$, $|\alpha| = 1$. Take $\alpha = 1$, $V = I_{n-1}$ to obtain $A = \begin{pmatrix} \sqrt{1+t^2} & 0 \\ 0 & I_{n-1} \end{pmatrix}$ and $d = \sqrt{1+t^2}$. Then g_W is given by (7) with $\lambda = t/\sqrt{1+t^2}$. The map in (9) is $\varphi_a = L^{-1} \circ f_{|a|} \circ L$, where $L \in U(n)$ with $La = (|a|, 0, \ldots, 0)$ and $f_{|a|}$ is the map in (7) with $\lambda = |a|$.

Let D be a bounded domain in $\mathbb{C}^n$. Consider the Bergman (Stefan –, 1895–1977) space $\mathscr{B}(D) = \mathscr{O}L^2(D)$. Let V_n (resp., σ_n) be the volume of B_n (resp., $S^{2n-1} := \partial B_n$).

Exercise 3.40. Let $f \in A(B_n(a,r))$. Then $f(a) = (V_n r^{2n})^{-1} \int_{B_n(a,r)} f(z)\, dV(z)$.

For $a \in D$ let $\delta_D(a) = d(a, \partial D)$, the distance from a to the boundary ∂D.

Lemma 3.41. *Let $f \in \mathscr{B}(D)$ and $a \in D$. Then $|f(a)| \le V_n^{-1/2}\delta_D(a)^{-n}\|f\|$.*

Lemma 3.42. *Let $u \in C(\overline{B}_{n-1})$. Let P be the projection $(z', z_n) \mapsto z'$. Then*

$$\int_{\partial B_n} u \circ P\, d\sigma = 2\pi \int_{B_{n-1}} u\, dV.$$

Proof. $d\sigma = (\cos\theta)^{-1} 2\pi |z_n| r dV = 2\pi dV$, θ being the angle between the normal direction and the $|z_n|$ direction. □

Exercise 3.43. Show that $\sigma_n = 2nV_n = 2\pi V_{n-1}$. Evaluate these numbers.

Theorem 3.44. *$\mathscr{B}(D)$ is a Hilbert space, and hence a closed subspace of $L^2(D)$.*

Proof. Let $\{f_j\}$ be a Cauchy sequence in $\mathscr{B}(D)$. Let $K \subset\subset D$. Choose $\delta > 0$ so that $\delta < dist(K, \partial D)$. For $a \in K$, we have

$$|f_j(a) - f_k(a)| \le (V_n r^{2n})^{-1/2} (\int_{B_n(a,r)} |f_j(z) - f_k(z)|\, dV(z))^{1/2} \le (V_n r^{2n})^{-1/2} \|f_j - f_k\|.$$

Thus f_j converges compactly to a holomorphic function f. For $\varepsilon > 0$, there is an N such that $\|f_j - f_k\| < \varepsilon$ whenever $j_j, k > N$. It follows that $\|f_j - f\|_K \le \varepsilon$ for each K. Thus $\|f_j - f\| \le \varepsilon$. Therefore $f \in \mathscr{B}(D)$ and $f_j \to f$. □

For $f, g \in \mathscr{B}(D)$, let $(f, g) = \int_D f(\zeta)\overline{g(\zeta)}\, d\zeta$. For $w \in D$, the evaluation map $\tau_w : \mathscr{B}(D) \to \mathbb{C}$ defined by $\tau_w(f) = f(w)$ is a bounded linear functional on $\mathscr{B}(D)$. By the Riesz representation theorem, there is a unique element in $\mathscr{B}(D)$, $K(\cdot, w)$, such that

$$f(w) = \tau_w(f) = (f, K(\cdot, w)), \quad f \in \mathscr{B}(D). \tag{10}$$

The function $K = K_D : D \times D \to \mathbb{C}$ is called the Bergman kernel for D.

Theorem 3.45. *Let $K(z, w)$ be the Bergman kernel for D. Then*
(i) $\|K(\cdot, w)\| \le V_n^{-1/2}\delta_D(w)^{-n}$,
(ii) $\overline{K(z, w)} = K(w, z)$.

Proof. (i) $\|K(\cdot,w)\| = \|\tau_w\| \le V_n^{-1/2}\delta_D(w)^{-n}$, by Lemma 3.41.
(ii) Replacing f in (10) by $K(\cdot,z)$ yields

$$K(w,z) = (K(\cdot,z), K(\cdot,w)).$$

Interchanging w,z gives

$$K(z,w) = (K(\cdot,w), K(\cdot,z)).$$

It follows that $\overline{K(z,w)} = K(w,z)$. □

So $K(z,w)$ is holomorphic in z and conjugate holomorphic in w.
Let $\{\varphi_j\}$ be an orthonormal basis for $\mathscr{B}(D)$.

Theorem 3.46. $K(z,w) = \sum \varphi_j(z)\overline{\varphi_j(w)}$.

Proof. Expand $K(\cdot,w)$ as a Fourier series and find the coefficients. □

Exercise 3.47. $K_{D\times\Omega}((z,\zeta),(w,\mu)) = K_D(z,w)\cdot K_\Omega(\zeta,\mu)$.

Theorem 3.48. *The transformation $f \mapsto Pf(z) = \int_D f(w)K(z,w)\,dV(w)$ is the orthogonal projection from $L^2(D)$ to $\mathscr{B}(D)$.*

Proof. We verify that $(f - Pf) \perp \mathscr{B}(D)$. □

Theorem 3.49. *Let f be a biholomorphic map from D onto Ω. Then*

$$K_D(z,w) = J_f(z)K_\Omega(f(z),f(w))\overline{J_f(w)}.$$

Proof. If $\{\psi_j\}$ is an orthonormal basis for Ω, then $\{J_f(z)\psi_j(f(z))\}$ is an orthonormal basis for D. □

Let $a_\alpha = \|z^\alpha\|_{\mathscr{B}(B_n)}$. Then $a_0 = \sqrt{V_n}$ and

$$K_{B_n} = \sum a_\alpha^{-2} z^\alpha \overline{w}^\alpha.$$

Theorem 3.50. $K_{B_n}(z,w) = V_n^{-1}(1 - \langle z,w\rangle)^{-n-1}$.

Proof. By the above formula, $K(0,0) = V_n^{-1}$. Consider the map in (7).We have $f(0) = p := (\lambda,0,\ldots,0)$, $J_f(0) = (1-\lambda^2)^{(n+1)/2}$. Thus $K(p,p) = V_n^{-1}(1-\lambda^2)^{-(n+1)}$. Let $z \in B_n$, $z \ne 0$. Since there is a unitary transformation U such that $Uz = (|z|,0,\ldots,0)$, it follows that $K(z,z) = V_n^{-1}(1-|z|^2)^{-(n+1)}$. Let $Q(z,w) = V_n^{-1}(1-\langle z,w\rangle)^{-n-1}$. Then $Q(z,z) = K(z,z)$. By Exercise 2.11, $Q(z,w) = K(z,w)$. □

Exercise 3.51. Let $\lambda \in (0,1)$ and let f_λ be the map in (7). Prove that there is a positive $u_\lambda \in C(\partial B_n)$ such that for each $h \in C(\partial B_n)$,

$$\int_{\partial B_n} h(f_\lambda(\zeta))\,d\sigma(\zeta) = \int_{\partial B_n} h(\zeta)u_\lambda(\zeta)\,d\sigma(\zeta).$$

Ans: $(1-\lambda^2)^n/|1-\lambda\zeta_1|^{2n}$.

Exercise 3.52. Let $\varphi \in Aut(B_n)$. Then there is a u_φ such that for each $h \in C(\partial B_n)$,

$$\int_{\partial B_n} h(\varphi(\zeta))\,d\sigma(\zeta) = \int_{\partial B_n} h(\zeta)u_\varphi(\zeta)\,d\sigma(\zeta).$$

Ans: $(1-|\varphi(0)|^2)^n/|1-\varphi(0)\cdot\overline{\zeta}|^{2n}$.

Let $\|\cdot\|$ denote the $L^2(\partial B_n)$ norm.

Proposition 3.53. *There is a function $q : [0,1) \to \mathbb{R}$ such that*

$$|f(z)| \le q(\|z\|)\|f\|, \quad f \in A(B_n). \tag{11}$$

Proof. Fix $a \in B_n$. Choose a $\varphi \in Aut(B_n)$ so that $\varphi(0) = a$. Then

$$|f(a)| = \sigma_n^{-1}|\int_{\partial B_n} f(\varphi(\zeta))\, d\sigma(\zeta)| = \sigma_n^{-1}|\int f(\zeta)u_\varphi(\zeta)\, d\sigma(\zeta)| \le \sigma_n^{-1}\|u_\varphi\|\cdot\|f\| = q(\|z\|)\|f\|.$$

□

Let $H^2(B_n)$ be the closure of $A(B)$ in $L^2(\partial B_n)$. By Proposition 3.53, each element in $H_2(B_m)$ is a holomorphic function on B_n. Similar to the above reasoning, there is a Cauchy kernel $C(z,w)$ so that for each $f \in H^2(B_n)$,

$$f(z) = \int_{\partial B_n} C(z,w)f(w)\, d\sigma(w).$$

Let $b_\alpha = \|z^\alpha\|_{L^2(\partial B_n)}$. Then

$$C(z,w) = \sum b_\alpha^{-2} z^\alpha \overline{w}^\alpha.$$

By Lemma 3.42, for $\alpha' \in \mathbb{N}^{n-1}$, $b_{(\alpha',0)} = \sqrt{2\pi}a_{\alpha'}$, where $a_{\alpha'} = \|(z')^{\alpha'}\|_{\mathscr{B}(B_{n-1})}$.

Theorem 3.54. *(L. Hua (Luogeng Hua, 1910–1985), 1958)* $C(z,w) = \sigma_n^{-1}(1 - z\cdot\overline{w})^n$.

Proof.

$$\begin{aligned} C((z',0),(w',w_n)) &= (2\pi)^{-1} \sum_{\alpha'\in\mathbb{N}^{n-1}} a_{\alpha'}^{-2}(z')^{\alpha'}(\overline{w}')^{\alpha'} \\ &= (2\pi)^{-1}K_{B_{n-1}}(z',w') = \sigma_n^{-1}(1 - z'\cdot\overline{w}')^{-n}. \end{aligned}$$

Since $C(Uz,Uw) = C(z,w)$ for each $U \in U(n)$, we obtain the desired formula. □

Exercise 3.55. Evaluate $\int_{B_n} \sin(x_1^2 y_2)K(w,z)\, dV(z)$.

Exercise 3.56. Evaluate $\int_{\partial B_n} \sin(x_1^2 y_2)C(w,z)\, d\sigma(z)$.

Exercise 3.57. Prove or disprove: $\int_{B_n} f(w)K(z,w)\, dV(w) = \int_{\partial B_n} f(w)C(z,w)\, d\sigma(w)$ for each $z \in B_n$ and each f real analytic in a neighborhood of $\overline{B}_n$.

4. Domains of Holomorphy

Let Ω be a domain in $\mathbb{C}^n$.

Definition 4.1. Let $f \in \mathscr{O}(\Omega)$ and $w \in \partial\Omega$. We say f *extends holomorphically beyond* w if there is a connected neighborhood W of w, a connected component U of $W \cap \Omega$ with $w \in \partial U$, and an $F \in \mathscr{O}(W)$ such that $f = F$ on U.

Remark. We do not require that $f = F$ on $W \cap \Omega$. That is because a boundary point may correspond to several reachable boundary points.

Definition 4.2. The domain Ω is said to be a *pointwise domain of holomorphy* if for each $w \in \partial\Omega$ there is an $f \in \mathscr{O}(\Omega)$ that does not extend holomophically beyond w.

Exercise 4.3. Each domain in $\mathbb{C}$ is a pointwise domain of holomorphy.

In $\mathbb{R}^N$ each real linear function ℓ can be written $\ell(x) = \langle x, v\rangle$, where $v \in \mathbb{R}^N$, and $\langle x, v\rangle := x_1v_1 + \cdots + x_Nv_N$ is the inner product. When $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$, we have $\langle z, \xi\rangle = \Re(z \cdot \overline{\xi}) = \Re(z_1\overline{\xi}_1 + \cdots + z_n\overline{\xi}_n)$.

The set of all real linear functions on $\mathbb{R}^N$ is denoted by $\mathscr{L}$. The linear hull of a compact set K is defined by

$$\hat{K}_{\mathscr{L}} := \{x \in \mathbb{R}^N : \ell(x) \le \max_{y\in K} \ell(y) \text{ for each } \ell \in \mathscr{L}\}.$$

Exercise 4.4. Show that $\hat{K}_{\mathscr{L}}$ is the convex hull of K, the intersection of all convex sets containing K.

If L is a line segment in $\mathbb{R}^N$, ∂L denotes the set of the two endpoints of L.

Exercise 4.5. Let Ω be a domain in $\mathbb{R}^N$. Then the following are equivalent.

(i) Ω is convex, *i.e.*, for $a, b \in \Omega$ the segment connecting a, b lies in Ω.

(ii) For each compact subset K of Ω, $\hat{K}_{\mathscr{L}} \subset \Omega$.

(iii) Whenever $\{L_\alpha\}$ is a family of line segment in Ω with $(\cup\partial L_\alpha)^- \subset \Omega$ one has $(\cup L_\alpha)^- \subset \Omega$.

(iv) For each line segment L in Ω, $d(L, \partial\Omega) = d(\partial L, \partial\Omega)$.

(v) For each $w \in \partial\Omega$, there is an $\ell \in \mathscr{L}$ such that $\ell(w) > \ell(z)$ for each $z \in \Omega$. □

Proposition 4.6. *Each convex domain is a pointwise domain of holomorphy.*

Proof. Let Ω be a convex domain and let $a \in \partial\Omega$. There is an $\ell \in \mathscr{L}$ such that $\ell(a) > \ell(z)$ for $z \in \Omega$. There is a $\xi \in \mathbb{C}^n$ with $\ell(z) = \Re(z \cdot \overline{\xi})$. Let $f(z) = ((z-a)\cdot\overline{\xi})^{-1}$. Then $f \in \mathscr{O}(\Omega)$ and f does not extend holomorphically beyond a. □

Definition 4.7. The domain Ω is said to be a *global domain of holomorphy* if there is an $f \in \mathscr{O}(D)$ that extends holomorphically beyond no boundary points.

Remark. $\mathbb{C}^n$ is trivially a global domain of holomorphy. A global domain of holomorphy is clearly a pointwise domain of holomorphy.

Let $\Delta := \Delta(0, 1) \subset \mathbb{C}$.

Exercise 4.8. Construct a Blaschke product on Δ that extends holomorphically beyond no boundary points. Hence Δ is a global domain of holomorphy.

Definition 4.9. The domain Ω is said to be a *domain of holomorphy* if there are no open sets D and U in $\mathbb{C}^n$ with the following properties.
(a) $\emptyset \neq D \subset U \cap \Omega$.
(b) U is connected and not contained in Ω.
(c) For each $f \in \mathscr{O}(\Omega)$ there is an $F \in \mathscr{O}(U)$ such that $f = F$ on D.

Proposition 4.10. *A pointwise domain of holomorphy is a domain of holomorphy.*

Proof. Exercise. □

Example 4.11. The Hartogs' domain is not a domain of holomorphy.

Definition 4.12. For a compact subset K of an open set $\Omega \subset \mathbb{C}^n$, the *holomorphic hull* of K in Ω is

$$\hat{K}_\Omega = \hat{K}_{\mathscr{O}(\Omega)} := \{z \in \Omega : |f(z)| \leq \|f\|_K \text{ for every } \mathscr{O}(D)\}.$$

Exercise 4.13. If $K \subset \Omega_1 \subset \Omega_2$ then $\hat{K}_{\Omega_1} \subset \hat{K}_{\Omega_2}$.

Exercise 4.14. Let Q be a bounded relatively closed subset of an open set $\Omega \subset \mathbb{R}^N$. Then the following are equivalent.
(i) Q is compact.
(ii) $\overline{Q} \subset \Omega$.
(iii) $d(Q, \partial\Omega) > 0$.

Definition 4.15. For a compact $K \subset \mathbb{C}^n$, the *polynomial hull* of K is

$$\hat{K} = \hat{K}_p := \{z \in \mathbb{C}^n : |f(z)| \leq \|f\|_K \text{ for every polynomial in } \mathbb{C}[z_1, \dots, z_n]\}.$$

Exercise 4.16. $\hat{K}_{\mathbb{C}^n} = \hat{K}_p$.

Definition 4.17. An open set $\Omega \subset \mathbb{C}^n$ is said to be *holomorphically convex* if for each compact $K \subset \Omega$, the set $\hat{K}_\Omega$ is compact.

Remark. The definition is intrinsic. It is clear that the biholomorphic image of a holomorphically convex domain is holomorphically convex. It does not follow directly from the definition that a biholomorphic image of a domain of holomorphy is a domain of holomorphy.

Exercise 4.18. Suppose that Ω and D are biholomorphically equivalent. Then Ω is holomorphically convex if and only if D is holomorphically convex.

Exercise 4.19. $\hat{K}_\Omega$ is bounded and relatively close in Ω.

Exercise 4.20. $\Omega \times D$ is holomorphically convex if Ω and D are.

Theorem 4.21. *(Cartan and Thullen 1932) (Peter Thullen 1907–1996) If a domain Ω is holomorphically convex then Ω is a global domain of holomorphy.*

Proof. Suppose that $\Omega \neq \mathbb{C}^n$. Let Q be a countable dense subset of Ω. Let $\{a_j\}$ be a sequence in Q so that each point in Q appears in the sequence infinitely many times. Let $\{K_j\}$ be a normal exhaustion of Ω. Let $r_j = d(x_j, \partial\Omega)$. Choose $w_j \in B_n(a_j, r_j) \setminus L_j$, where L_j is the $\mathscr{O}(\Omega)$-hull of $(K_j \cup \{w_1, \dots, w_{j-1}\})$. There is an $f_j \in \mathscr{O}(\Omega)$ so that $f_j(w_j) =$

$j+1-\sum_{k=1}^{j-1} f_k(w_j)$ and $\|f_j\|_{L_j} < 2^{-j}$. Let $f = \sum f_j$. Then $f(w_j) = j+1+\sum_{k=j+1}^{\infty} f_k(w_j)$ and therefore $|f(w_j)| > j$.

Suppose that f extends holomorphically beyond $w \in \partial\Omega$. So there is a domain W containing w, an $F \in \mathscr{O}(W)$, and a connected component U of $\Omega \cap W$ with $w \in \partial U$ such that $f = F$ on U. There is a subsequence $\{a_{j_\nu}\}$ in U with $B_n(a_{j_\nu}, r_{j_\nu}) \subset U$ and $a_{j_\nu} \to w$. It follows that $w_{j_\nu} \in U$ and $w_{j_\nu} \to w$. Then $F(w) = \lim f(w_{j_\nu})$ and $|f(w_{j_\nu})| \to \infty$, which is absurd.. Therefore f extends holomorphically beyond no boundary points. □

Theorem 4.22. *(Oka 1936) (Kiyoshi Oka 1901–1978) Let K be a compact set in a domain $\Omega \subset \mathbb{C}^n$, let D be a logarithmically convex complete Reinhardt domain in $\mathbb{C}^n$, and let $f, g \in \mathscr{O}(D)$. Suppose that for each $a \in K$ we have $(a + f(a)D) \subset \Omega$. Then for each $b \in \hat{K}_\Omega$, the Taylor series of g at b converges normally in $b + f(b)D$.*

Proof. Let $\Gamma := \overline{\Delta}^n(0, r)$ be a compact polydisc in D. Let $Q = \{a + f(a)z : a \in K, z \in \Gamma\}$ and $M = \|g\|_Q$. Then

$$|f(a)^{|\alpha|} D^\alpha g(a)| \le M\alpha! r^{-\alpha}, \quad \text{for } \alpha \in \mathbb{N}^n \ a \in K.$$

Thus

$$|f(b)^{|\alpha|} D^\alpha g(b)| \le M\alpha! r^{-\alpha}, \quad \text{for } \alpha \in \mathbb{N}^n, \ b \in \hat{K}_\Omega.$$

This implies that the Taylor series of g at b converges normally in $b + f(b)(\operatorname{int}\Gamma)$ for each such Γ, and therefore in $b + f(b)D$. □

Corollary 4.23. *Let Ω be a domain in $\mathbb{C}^n$ and let D be a logarithmically convex complete Reinhardt domain in $\mathbb{C}^n$. Let K be a compact subset of Ω and let $d > 0$. Suppose that $K + dD \subset \Omega$. Then every function in $\mathscr{O}(\Omega)$ extends holomorphically to $\hat{K}_\Omega + dD$.*

Fix a bounded logarithmically convex complete Reinhardt domain D in $\mathbb{C}^n$. Let $\delta_D(K, \partial\Omega)$ denote the largest $r > 0$ so that $K + rD \subset \Omega$.

Corollary 4.24. *Let Ω be a domain of holomorphy and let K be a compact subset of Ω. Then $\delta_D(K, \partial\Omega) = \delta_D(\hat{K}_\Omega, \partial\Omega)$.*

Theorem 4.25. *Let Ω be a domain in $\mathbb{C}^n$. Then the following are equivalent.*
(i) Ω is a global domain of holomorphy.
(ii) Ω is a pointwise domain of holomorphy.
(iii) Ω is a domain of holomorphy.
(iv) For each compact subset K of Ω, $\delta_D(K, \partial\Omega) = \delta_D(\hat{K}_\Omega, \partial\Omega)$.
(v) Ω is holomorphically convex.

Replacing “domain” with “open set”, we similarly define open set of holomorphy, global open set of holomorphy, and pointwise open set of holomorphy. It is clear that an open set is an open set of holomorphy if and only if each of its connected component is a domain of holomorphy, and that Theorem 4.25 remains true when “domain” is replaced with “open set”.

Theorem 4.26. *Let $\{\Omega_\alpha\}$ be a family of holomorphically convex open sets. Suppose that the interior Ω of the intersection $\cap\Omega_\alpha$ is nonempty. Then Ω is holomorphically convex.*

Proof. Let K be a compact subset of Ω and let $r = d(K, \partial\Omega)$. Then for each α, $\hat{K}_\Omega \subset \hat{K}_{\Omega_\alpha}$, and hence $d(\hat{K}_\Omega, \partial\Omega_\alpha) \geq d(\hat{K}_{\Omega_\alpha}, \partial\Omega_\alpha) = d(K, \partial\Omega_\alpha) \geq r$. It follows that $d(\hat{K}_\Omega, \partial\Omega) = \inf_\alpha d(\hat{K}_\Omega, \partial\Omega_\alpha) \geq r$ and $\hat{K}_\Omega \subset\subset \Omega$. Thus Ω is holomorphically convex. □

Lemma 4.27. *Let Ω be a logarithmically convex complete Reinhardt domain. Let K be a compact set in Ω. Then $\hat{K}_p \subset \Omega$.*

Proof. Choose a finite set S of $r \in \mathbb{R}^n$ with $r > 0$ such that $K \subset \Gamma := \cup_{r \in S} \overline{\Delta}^n(0, r) \subset \Omega$. Let Q be the logarithmically convex hull of Γ. Then $Q \subset \Omega$. It suffices to show that $\hat{\Gamma}_p \subset Q$. Suppose that $z \in \hat{\Gamma}_p$. Then $|z^\alpha| \leq \max_{r \in S} r^\alpha$ for all $\alpha \in \mathbb{N}^n$. It follows that

$$|z_1|^{s_1} \cdots |z_n|^{s_n} \leq \max_{r \in S} r_1^{s_1} \cdots r_n^{s_n} \text{ for } (s_1, \ldots, s_n) \in \mathbb{R}^n_+.$$

This implies that $z \in Q$. Thus $\hat{\Gamma}_p \subset Q$. □

Theorem 4.28. *Let Ω be a Reinhardt domain containing 0. Then the following are equivalent.*

(i) There is a power series $g \in \mathbb{C}\{z\}$ with $\Omega = \mathscr{Q}(g)$.
(ii) Ω is a domain of holomorphy.
(iii) Ω is a logarithmically convex complete Reinhardt domain.

Proof. (i) ⇒ (iii): Theorem 3.13.

(iii) ⇒ (ii): If Ω is a logarithmically convex complete Reinhardt domain and K is compactum in Ω then Lemma 4.27 tells us that $\hat{K}_\Omega \subset \hat{K}_p \subset\subset \Omega$, hence Ω is a domain of holomorphy.

(ii) ⇒ (i): Suppose that (ii) holds. Then there is an $f \in \mathscr{O}(\Omega)$ that extends holomorphically beyond no boundary points. By Theorem 3.30, $\Omega \subset \mathscr{Q}(g)$, where g is the Taylor series of f at 0. Since f extends holomorphically beyond no boundary points, we must have $\Omega = \mathscr{Q}(g)$. □

Definition 4.29. A *tube domain* in $\mathbb{C}^n$ is a domain Ω of the form $\Omega = \omega + i\mathbb{R}^n = \{z \in \mathbb{C}^n : \Re z \in \omega\}$, where ω is a domain in $\mathbb{R}^n$. The domain ω is called *the base* of Ω.

For $\varepsilon \in (0, 1/2)$, set

$$\begin{aligned} k &= \{(x_1, x_2, 0, \ldots, 0) : 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1, x_1 x_2 = 0\} \\ K &= \{(x_1, x_2, 0, \ldots, 0) : 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1\} \\ k_\varepsilon &= \{x + iy : x \in k, y_1^2 + y_2^2 \leq 1/\varepsilon, y_3 = \cdots = y_n = 0\} \\ K_\varepsilon &= \{x \in K : x_1 + x_2 - \varepsilon(x_1^2 + x_2) \leq 1 - \varepsilon\}. \end{aligned}$$

Lemma 4.30. *Let Ω be a tube domain containing $K + i\mathbb{R}^n$, and let $\varepsilon \in (0, 1/2)$. Then the $\mathscr{O}(\Omega)$-hull of k_ε contains K_ε and $(1 - \varepsilon)K$.*

Proof. Let $\Gamma = \{(y_1, y_2, 0, \ldots, 0) : (y_1, y_2) \in \mathbb{R}^2\}$. Consider $h(z) := z_1 + z_2 - \varepsilon(z_1^2 + z_2^2) - (1 - \varepsilon)$. Let $M = \{z \in (K + i\Gamma) : h(z) = 0\}$. We have in M,

$$x_1 + x_2 - \varepsilon(x_1^2 + x_2^2) + \varepsilon(y_1^2 + y_2^2) = 1 - \varepsilon.$$

Hence $x_1^2 + x_2^2 \leq 1$, $y_1^2 + y_2 \leq 1/\varepsilon$, so M is compact. (If $x_1^2 + x_2^2 > 1$, then the left side $> (x_1^2 + x_2^2) - \varepsilon(x_1^2 + x_2^2) > 1 - \varepsilon$.) Since $x_1 + x_2 < 1$ on M except at the points $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$, the boundary of M belongs to k_ε. It follows that the $\mathscr{O}(\Omega)$-hull of k_ε

contains M. Thus the hull contains $M \cap (K + i0) = \partial K_\varepsilon$. For $\lambda \in (0, 1)$, by considering $h(z/\lambda)$ we see that the hull contains $\lambda \partial K_\varepsilon$. Therefore the hull contains K_ε. Finally, the hull contains $(1-\varepsilon)K$ because $(1-\varepsilon)K \subset K_\varepsilon$. ☐

For a domain $Q \subset \mathbb{R}^N$, let $\hat{Q}$ denote the convex hull of Q, the intersection of all convex domains containing Q.

Definition 4.31. A set $Q \subset \mathbb{R}^N$ is said to be *starlike* with respect to a point a if whenever $p \in Q$, the line segment connecting a and p lies in Q.

Theorem 4.32. *(S. Bochner 1943) (Saloman Bochner 1899–1982) Let Ω be a tube domain. Then every $u \in \mathscr{O}(\Omega)$ can be extended to a function in $\mathscr{O}(\hat{\Omega})$.*

Proof. (a) We first assume that the base ω is starlike with respect to 0. There is a tube $\tilde{\Omega}$ with a base $\tilde{\omega}$ that is starlike with respect to 0 such that every $g \in \mathscr{O}(\Omega)$ extends holomorphically to $\tilde{\Omega}$, and $\tilde{\Omega}$ contains every tube that is starlike with respect to 0 with this property. In fact, we need only take the union of all such tubes.

We have to show that $\tilde{\omega}$ is convex. Let $v_1, v_2 \in \tilde{\omega}$ be nonzero. Set

$$\Gamma = \{\lambda_1 v_1 + \lambda_2 v_2 : 0 \le \lambda_1, 0 \le \lambda_2, \lambda_1 + \lambda_2 \le 1\},$$
$$\Lambda = \{\lambda_1 v_1 + \lambda_2 v_2 : 0 \le \lambda_1, 0 \le \lambda_2, \lambda_1 + \lambda_2 \le 1, \lambda_1\lambda_2 = 0\},$$
$$E = \{a : 0 \le a \le 1, a\Gamma \subset \tilde{\omega}\}.$$

Let $\mu = \max(|v_1|, |v_2|)$. Let d be the distance from Λ to $\partial\tilde{\omega}$. Let $s = \sup E$ and choose α, β so that $\max(0, s - d(2\mu)^{-1}) < \alpha < \beta < s$. Then $\beta \in E$. By Lemma 4.30, there is a compact set $Q \in \mathbb{R}^n$ such that the $\mathscr{O}(\tilde{\Omega})$-hull of $\Lambda + iQ$ contains $\alpha\Gamma$. Since $(\Lambda + iQ) + dB_n$ lies in $\tilde{\Omega}$, it follows from Corollary 4.23 that every function in $\mathscr{O}(\Omega)$ extends holomorphically to $(\alpha\Gamma + i0) + dB_n$. Let $\Gamma_{\alpha,d}$ be the d-neighborhood of $\alpha\Gamma$ in $\mathbb{R}^n$. Then every function in $\mathscr{O}(\Omega)$ extends holomorphically to $\Gamma_{\alpha,d} + i\mathbb{R}^n$. Since $\tilde{\omega} \cup \Gamma_{\alpha,d}$ is starlike with respect to 0, we see that $\Gamma_{\alpha,d} \subset \tilde{\omega}$. Let $b = \min(1-\alpha, d(2\mu)^{-1})$. We now show that $\alpha + b \in E$. Suppose that $y = (\alpha + b)x$ with $x \in \Gamma$. Then $|x| \le \mu$ and $d(y, \alpha\Gamma) \le d(y, \alpha x) = b|x| \le b\mu \le d(2\mu)^{-1}\mu = d/2$, and hence $y \in \Gamma_{\alpha,d} \subset \tilde{\omega}$. It follows that $(\alpha + b)\Gamma \subset \tilde{\omega}$ and $(\alpha + b) \in E$. Since $\alpha + b \le s < \alpha + d(2\mu)^{-1}$, we see that $b < d(2\mu)^{-1}$ and hence $b = 1 - \alpha$. Therefore $1 \in E$ and $\Gamma \in \tilde{\omega}$ and $\tilde{\omega}$ is convex. We have proved that if ω is starlike with respect to 0 then every function in $\mathscr{O}(\Omega)$ extends holomorphically to a function in $\mathscr{O}(\tilde{\Omega})$.

(b) Now let ω be a domain in $\mathbb{R}^n$. Let $0 \in \omega$ and denote by $\tilde{\omega}$ the largest tube whose base is starlike with respect to 0 such that for each $f \in \mathscr{O}(\Omega)$ there is an $\tilde{f} \in \mathscr{O}(\tilde{\Omega})$ so that $f = \tilde{f}$ in a neighborhood of 0. According to (a), $\tilde{\Omega}$ is convex.

Now assume that $\tilde{\Omega}$ does not contain Ω. There is an $x_0 \in \omega \setminus \tilde{\omega}$. We can join x_0 to 0 with a polygon in ω. Let x_1 be its last intersection with $\partial\tilde{\omega}$. Let ω_1 be a small ball centered at x_1 and contained in ω. Then f and $\tilde{f}$ coincide in $(\omega_1 \cap \tilde{\omega}) + i\mathbb{R}^n$. Thus the function $\tilde{f}$ extends holomorphically to a tube with base $\tilde{\omega} \cup \omega_1$, which is starlike with respect to 0. This contradicts the defintion of $\tilde{\Omega}$. Hence $\tilde{\Omega} \supset \Omega$. ☐

Theorem 4.33. *Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^m$ be domains of holomorphy. and let $h \in \mathscr{O}(\Omega, \mathbb{C}^n)$. Then $Q := h^{-1}(D)$ is a domain of holomorphy.*

Proof. Let K be a compact set in Q. Since $\hat{K}_Q \subset \hat{K}_\Omega \subset\subset \Omega$, the closure Γ of $\hat{K}_Q$ lies in Ω. Since $h(\hat{K}_Q) \subset \widehat{(h(K))}_D$, we see that $h(\Gamma) \subset \widehat{(h(K))}_D$ and hence $\Gamma \subset Q$. ☐

Corollary 4.34. *Let Ω be a domain of holomorphy and let $f_1, \dots, f_N \in \mathscr{O}(\Omega)$. Then $\Omega_f := \{z \in \Omega : |f_j(z)| < 1, j = 1, \dots, N\}$ is a domain of holomorphy.*

Theorem 4.35. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $D \subset \mathbb{C}^m$ be a domain of holomorphy. Let $h \in \mathscr{O}(\Omega, \mathbb{C}^n)$. Suppose that $Q := h^{-1}(D) \subset\subset \Omega$. Then Q is a domain of holomorphy.*

Proof. Let K be a compact set in Q. Since $\hat{K}_Q \subset Q \subset\subset \Omega$, the closure Γ of $\hat{K}_Q$ lies in Ω. Since $h(\hat{K}_Q) \subset \widehat{(h(K))}_D$, we see that $h(\Gamma) \subset \widehat{(h(K))}_D$ and hence $\Gamma \subset Q$. □

Theorem 4.36. *Let $\omega \subset \mathbb{C}$ be a domain with C^1 boundary. If $u \in C^1(\overline{\omega})$, we have*

$$u(\zeta) = (2\pi i)^{-1}\Big(\int_{\partial\omega} \frac{u(z)\,dz}{z-\zeta} + \iint_{\omega} \frac{\partial u/\partial \overline{z}}{z-\zeta} dz \wedge d\overline{z}\Big),\ \zeta \in \omega.$$

Proof. Let $\zeta \in \omega$. Assume $\varepsilon > 0$ and $\overline{\Delta}(\zeta, \varepsilon) \subset \omega$. By Stokes' Theorem,

$$\begin{aligned} -\iint_{\omega - \overline{\Delta}(\zeta,r)} \frac{\partial u/\partial \overline{z}}{z-\zeta} dz \wedge d\overline{z} &= \int_{\partial\omega} \frac{u(z)\,dz}{z-\zeta} - \int_{|z-\zeta|=\varepsilon} \frac{u(z)\,dz}{z-\zeta} \\ &= \int_{\partial\omega} \frac{u(z)\,dz}{z-\zeta} - i\int_0^{2\pi} u(\zeta + \varepsilon e^{it})\,dt. \end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$-\iint_{\omega} \frac{\partial u/\partial \overline{z}}{z-\zeta} dz \wedge d\overline{z} = \int_{\partial\omega} \frac{u(z)\,dz}{z-\zeta} - 2\pi i u(\zeta).$$

□

Theorem 4.37. *Let $\varphi \in C^1_0(\mathbb{C})$ and $u(\zeta) := (2\pi i)^{-1} \iint \varphi(z)(z-\zeta)^{-1}\,dz \wedge d\overline{z}$. Then $\partial u/\partial \overline{z} = \varphi$.*

Proof. We have $u(\zeta) = -(2\pi i)^{-1} \iint \varphi(\zeta - z) z^{-1}\,dz \wedge d\overline{z}$. Thus

$$\begin{aligned} \partial u/\partial \overline{\zeta} &= -(2\pi i)^{-1} \iint \partial\varphi(\zeta - z)/\partial\overline{\zeta}\ z^{-1}\,dz \wedge d\overline{z} \\ &= (2\pi i)^{-1} \iint \partial\varphi(z)/\partial\overline{z}\ (z-\zeta)^{-1}\,dz \wedge d\overline{z} = \varphi(\zeta). \end{aligned}$$

The last step uses the previous theorem. □

Exercise 4.38. Show that $\partial/\partial\overline{z}(1/z) = \pi\delta_0(z)$. Prove that $\partial^2/\partial z \partial\overline{z} = (1/4)\Delta$ and $\Delta \log|z| = 2\pi\delta_0(z)$.

Exercise 4.39. Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy. Show that there is a normal exhaustion $\{K_j\}$ so that $\hat{K}_j = K_j$.

Theorem 4.40. *Let $\Omega \subset \mathbb{C}$ and $f \in C^\infty(\Omega)$. Then there is a $u \in C^\infty(\Omega)$ such that $\partial u/\partial \overline{z} = f$.*

Proof. Let $\{K_j\}$ be a normal exhaustion of Ω with $\hat{K}_j = K_j$. There exist $\psi_j \in C_0^\infty(\operatorname{int} K_{j+1})$, $0 \le \psi_j \le 1$, $\psi_j = 1$ in some neighborhood of K_j. Let $\varphi_j = \psi_j - \psi_{j-1}$. Then $\sum \varphi_j = 1$. We solve $\partial u_j/\partial\overline{z} = \varphi_j f$ to get u_j. Then u_j is holomorphic in some neighborhood of K_{j-1}. By Runge's theorem, there are $v_j \in \mathscr{O}(\Omega)$ so that $\|u_j - v_j\|_{K_{j-1}} \le 2^{-j}$. Let $u = \sum(u_j - v_j)$. □

4.41. Differential forms and Dolbeault cohomology.

Differential forms of the form

$$f = \sum_{|I|=p,|J|=q} f_{I,J} dz^I \wedge d\overline{z}^J,$$

where the sum is taken increasing I and J, $f_{I,J} \in C^\infty(\Omega)$, are called C^∞ (p,q)-forms. The set of such forms is denoted by $C^\infty_{p,q}(\Omega)$. From $d^2 = 0$ we derive that $\overline{\partial}^2 = 0$. Let $Z_{p,q}$ and $B_{p,q}$ denote the $\overline{\partial}$-closed and $\overline{\partial}$-exact forms:

$$Z_{p,q}(\Omega) = \{f \in C^\infty_{p,q}(\Omega) : \overline{\partial} f = 0\}, \quad B_{p,q}(\Omega) = \{\overline{\partial} g : g \in C^\infty_{p,q-1}(\Omega)\}.$$

Dolbeault cohomology groups are defined by

$$H^{p,q}_{\overline{\partial}}(\Omega) = \frac{Z_{p,q}(\Omega)}{B_{p,q}(\Omega)}.$$

We write $H^{0,q}_{\overline{\partial}}(\Omega) = H^q_{\overline{\partial}}(\Omega)$.

Exercise 4.42. (i) If $h : \Omega \to D$ holomorphic, then h^* commutes with $\overline{\partial}$.
(ii) $H^0_{\overline{\partial}}(\Omega) = \mathscr{O}(\Omega)$.
(iii) $H^q_{\overline{\partial}}(\Omega) = 0$ for $q > n$.
(iv) If $H^q_{\overline{\partial}}(\Omega) = 0$, then $H^{p,q}_{\overline{\partial}}(\Omega) = 0$ for all p.

Proposition 4.43. *Let $\Omega \subset \mathbb{C}$. Then $H^1_{\overline{\partial}}(\Omega) = 0$.*

This is a consequence of Theorem 4.40.

Exercise 4.44. Let $\Omega \subset \mathbb{C}^n$. Then $H^n_{\overline{\partial}}(\Omega) = 0$.

Hint: By theory of PDE the equation $f = \Delta g$ can be solved.

Theorem 4.45. *Let $f = \sum f_j \, d\overline{z}_j$ be a $\overline{\partial}$-closed $(0,1)$-form with $f_j \in C^\infty_0(\mathbb{C}^n)$ and $n > 1$. Then there is a $u \in C^\infty_0(\mathbb{C}^n)$ such that $\overline{\partial} u = f$.*

Remark. The theorem fails when $n = 1$. Suppose that $g \in C^\infty_0(\mathbb{C})$ with $\int_{\mathbb{C}} g \, dz \wedge d\overline{z} = 1$. If $u \in C^\infty_0(\mathbb{C})$ and $\overline{\partial} u = g d\overline{z}$, then $1 = \int_{\mathbb{C}} g \, dz \wedge d\overline{z} = -\int_{\mathbb{C}} d(u \, dz) = 0$.

Proof. Let

$$\begin{aligned} u(\zeta) :=& (2\pi i)^{-1} \int f_1(z_1, \zeta_2, \dots, \zeta_n)(z_1 - \zeta_1)^{-1} \, dz_1 \wedge d\overline{z}_1 \\ =& -(2\pi i)^{-1} \int f_1(\zeta_1 - z_1, \zeta_2, \dots, \zeta_n)(z_1)^{-1} \, dz_1 \wedge d\overline{z}_1. \end{aligned}$$

Then $u(\zeta) = 0$ when $|\zeta_2| + \cdots + |\zeta_n|$ is sufficiently large. The second form shows that u is C^∞. We have $\partial u / \partial \overline{\zeta}_1 = f_1$ by Theorem 4.37. For $k > 1$ we have

$$\begin{aligned} \frac{\partial u}{\partial \overline{\zeta}_k}(\zeta) =& -(2\pi i)^{-1} \int \frac{\partial f_1}{\partial \overline{\zeta}_k}(\zeta_1 - z_1, \zeta_2, \dots, \zeta_n)(z_1)^{-1} \, dz_1 \wedge d\overline{z}_1 \\ =& -(2\pi i)^{-1} \int \frac{\partial f_k}{\partial \overline{\zeta}_1}(\zeta_1 - z_1, \zeta_2, \dots, \zeta_n)(z_1)^{-1} \, dz_1 \wedge d\overline{z}_1 = f_k(\zeta). \end{aligned}$$

Thus $\overline{\partial} u = f$. It follows that u is holomorphic outside a compact set. Since $u(\zeta) = 0$ when $|\zeta_2| + \cdots + |\zeta_n|$ is sufficiently large, it follows from the identity theorem that $u = 0$ outside a compact set. $\square$

Theorem 4.46. *Let $n > 1$ and let $\Omega \subset \mathbb{C}^n$ be a domain. Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. Then for each $f \in \mathscr{O}(\Omega \setminus K)$ there is an $F \in \mathscr{O}(\Omega)$ so that $f = F$ on $\Omega \setminus K$.*

Proof. Let $h \in C_0^\infty(\Omega)$, $0 \le h \le 1$, and $h = 1$ in a neighborhood of K. Let Γ be the support of h. Let $g = \overline{\partial}(f \cdot (1 - h))$. Then g is a C^∞ $(0,1)$-form in Ω with $g = 0$ outside Γ. So g extends to be a member of $D_{(0,1)}(\mathbb{C}^n)$, the set of $(0,1)$ forms whose coefficients belong to $C_0^\infty(\mathbb{C}^n)$. There is a $u \in C_0^\infty(\mathbb{C}^n)$ such that $\overline{\partial} u = g$. The function u is zero in the unbounded component U of $\mathbb{C}^n \setminus \Gamma$. Let $F = f \cdot (1 - h) - u$. Then $F \in \mathscr{O}(\Omega)$. Since $F = f$ in $U \cap \Omega$, it follows that $F = f$ in $\Omega \setminus K$ since $\Omega \setminus K$ is connected. $\square$

Cochain, cocycle, and coboundary.

Theorem 4.47. *Let $\Omega \subset \mathbb{C}^n$ be a domain with $H^1_{\overline{\partial}}(\Omega) = 0$. For a one-cochain $g_{jk} \in \mathscr{O}(\Omega_j \cap \Omega_k)$ with $g_{jk} + g_{k\ell} + g_{\ell j} = 0$, we can find g_j so that $g_{jk} = g_k - g_j$.*

(Cousin I problem is solvable on Ω.)

Proof. There exists a locally finite partition of unity $0 \le \varphi_\nu \in C_0^\infty(\Omega_{i_\nu})$, $\sum \varphi_\nu = 1$. Let $h_k = \sum \varphi_\nu g_{i_\nu k}$. Then $h_k - h_j = g_{jk}$. This implies $\overline{\partial} h_k = \overline{\partial} h_j$ on $\Omega_j \cap \Omega_k$, globally defining a $\overline{\partial}$-closed $(0,1)$-form ψ. There is a u so that $\overline{\partial} u = \psi$. Let $g_k = h_k - u$. $\square$

Example 4.48. Let $\Omega = \mathbb{C}^2 \setminus \{0\}$. Then $H^1_{\overline{\partial}}(\Omega) \neq 0$.

Proof. Let $\Omega_1 = \{z_1 \neq 0\}$, $\Omega_2 = \{z_2 \neq 0\}$. Let $\psi(z) = 1/(z_1^k z_2^\ell)$, where $k, \ell > 0$. Then $\psi \in \mathscr{O}(\Omega_1 \cap \Omega_2)$. We claim that there are no $g_j \in \mathscr{O}(\Omega_j)$, $j = 1, 2$, such that $\psi = g_2 - g_1$ on $\Omega_1 \cap \Omega_2$. Suppose,if possible, that such g_1, g_2 exist. Then

$$
\begin{aligned}
1 &= \frac{1}{2\pi i} \int_{|z_1|=1} \int_{|z_2|=1} z_1^{k-1} z_2^{\ell-1} \psi(z)\, dz_1 dz_2 \\
&= \frac{1}{2\pi i} \int_{|z_2|=1} (\int_{|z_1|=1} z_1^{k-1} z_2^{\ell-1} g_2(z)\, dz_1) dz_2 - \frac{1}{2\pi i} \int_{|z_1|=1} (\int_{|z_2|=1} z_1^{k-1} z_2^{\ell-1} g_1(z)\, dz_2) dz_1 = 0.
\end{aligned}
$$

Let $h_j \in C^\infty(\Omega)$ be such that $h_j = 0$ in a neighborhood of $\{z_j = 0\}$ in Ω, and $h_1 + h_2 = 1$ in Ω. Let

$$
\psi_1 = -\psi(z) h_2(z), \quad \psi_2 = \psi(z) h_1(z),
$$

and $\varphi_j = \overline{\partial} \psi_j$. Then $\varphi_j \in C^\infty_{(0,1)}(\Omega_j)$. On $\Omega_1 \cap \Omega_2$, $\varphi_2 - \varphi_1 = \overline{\partial} \psi = 0$. Let $\varphi = \varphi_j$ on Ω_j. Then φ is a $\overline{\partial}$-closed form on Ω. We shall prove that φ is not a $\overline{\partial}$-exact form. Suppose that $\varphi = \overline{\partial} u$ with $u \in C^\infty(\Omega)$. Let $g_j = \psi_j - u$. Then $\overline{\partial} g_j = 0$ and $g_j \in \mathscr{O}(\Omega_j)$. It follows that $g_2 - g_1 = \psi$, contradicting the assertion that such g_1, g_2 do not exist. $\square$

In fact, we will see later that $H^q_{\overline{\partial}}(\mathbb{C}^n - \{0\}) = 0$ for $1 \le q \le n - 2$.

Exercise 4.49. Let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{z \in \mathbb{C}^2 : |z_1| < 1/2, |z_2| < 1\}$, $\Omega_2 = \{|z_1| < 1, 1/2 < |z_2| < 1\}$. Prove that $H^1_{\overline{\partial}}(\Omega) \neq 0$.

Hint: Imitate the example with $\psi = (z_1 - 3/4)^{-k} z_2^{-\ell}$. Integrate on $|z_j| = 7/8$.

For $z \in \mathbb{C}^n$, let $\mathscr{M}_z$ be the quotient field of $\mathscr{O}_{n,z}$.

Definition 4.50. A *meromorphic function* φ in a domain $\Omega \subset \mathbb{C}^n$ is a map

$$\varphi : \ \Omega \to \cup_{z \in \Omega} \mathscr{M}_z$$

such that $\varphi(z) \in \mathscr{M}_z$ for each z, and for every point a in Ω there is a neighborhood ω of a and functions $f, g \in \mathscr{O}(\omega)$ such that $\varphi(z) = f_z/g_z$ when $z \in \omega$. The set of all meromorphic functions in Ω is denoted by $\mathscr{M}(\Omega)$.

If $\Omega \subset \mathbb{C}$ and $\varphi \in \mathscr{M}(\Omega)$ then $\varphi = f/g$ for some $f, g \in \mathscr{O}(\Omega)$. That is a fact in complex analysis of one variable.

The problem whether every $\varphi \in \mathscr{M}(\Omega)$ has a global quotient representation $\varphi = f/g$, with $f, g \in \mathscr{O}(\Omega)$ was called the Poincaré problem. Poincaré gave an affirmative answer for $\Omega = \mathbb{C}^2$ in 1888. K. Oka gave a counterexample for $\Omega = Q^2$, where $Q = \{z \in \mathbb{C} : 3/4 < |z| < 5/4\}$, in 1939.

There exists a meromorphic function that does not have a global quotient representation if and only if there is a nontrivial line bundle such that dim(holo sections) greater than 1. In that case there are two linearly independent holomorphic sections u, v of the line bundle. Then u/v is a meromorphic function without global quotient representation.

Theorem 4.51. *(Mittag-Leffler Theorem) Let $\Omega \subset \mathbb{C}^n$ be a domain with $H^1_{\overline{\partial}}(\Omega) = 0$. Suppose that $\Omega = \cup \Omega_j$, $f_j \in \mathscr{M}(\Omega_j)$ with $f_j - f_k \in \mathscr{O}(\Omega_j \cap \Omega_k)$. Then we can find a meromorphic function in Ω so that $f - f_j \in \mathscr{O}(\Omega_j)$.*

(Locally given principal parts can be pieced together to get a global meromorphic function.)

Proof. Let $g_{jk} = f_k - f_j$. Then $\{g_{jk}\}$ is a "one-cocycle". By Theorem 4.47, there exists $\{g_j\}$ so that $g_{jk} = g_k - g_j$. Hence $f_j - g_j = f_k - g_k$ on $\Omega_k \cap \Omega_j$, globally defining a meromorphic function f. □

Theorem 4.52. *$H^{p,q}_{\overline{\partial}}(\Delta^n) = 0$ for $q > 0$.*

Proof. See Gunning and Rossi, page 28. □

Lemma 4.53. *Let $q \geq 0$, $n > 1$, and let $\Omega \subset \mathbb{C}^n$ be an open set with $H^{q+1}_{\overline{\partial}}(\Omega) = 0$. Suppose that $D := \{z_1 = 0\} \cap \Omega \neq \emptyset$. Let $\iota : D \to \Omega$ be the inclusion map. Then for each $\overline{\partial}$-closed $(0, q)$ form f on D there is a $\overline{\partial}$-closed $(0, q)$ form F on Ω with $f = \iota^* F$.*

Proof. Let $\pi : \Omega \to \{z_1 = 0\}$ be the projection map that drops the first coordinate. The sets D and $\Omega \setminus \pi^{-1}(D)$ are disjoint relatively closed sets in Ω. So there is an $h \in C^\infty(\Omega)$ so that $h = 1$ on D and $h = 0$ on $\Omega \setminus \pi^{-1}(D)$. The form $g := h \cdot \pi^* f$ is a well defined C^∞ $(0, q)$ form on Ω. Then $\overline{\partial} g$ is $\overline{\partial}$-closed on Ω and it vanishes in a neighborhood of D. Thus $z_1^{-1} \overline{\partial} g$ is a $\overline{\partial}$-closed $(0, q+1)$ form on Ω. Since $H^{q+1}_{\overline{\partial}}(\Omega) = 0$, there is a $(0, q)$ form u such that $\overline{\partial} u = z_1^{-1} \overline{\partial} g$. Let $F = g - z_1 u$. □

Theorem 4.54. *Let $\Omega \subset \mathbb{C}^n$ be an open set with $n > 1$. Let $D = \Omega \cap \{z_1 = 0\}$. Let $q \geq 1$. Suppose that $H^q_{\overline{\partial}}(\Omega) = H^{q+1}_{\overline{\partial}}(\Omega) = 0$. Then $H^q_{\overline{\partial}}(D) = 0$.*

Proof. Let $f \in Z_{0,q}(D)$. By Lemma 4.53, there is an $F \in Z_{0,q}(\Omega)$ with $f = \iota^* F$. Since $H^q_{\overline{\partial}}(\Omega) = 0$, there is a $U \in C^\infty_{0,q-1}(\Omega)$ such that $F = \overline{\partial} U$. Let $u = \iota^* U$. Then $f = \overline{\partial} u$. $\square$

Suppose that $\emptyset \neq D \subset U \cap \Omega$ and $f \in \mathscr{O}(\Omega)$. We say that f extends holomorphically from D to U if there is an $F \in \mathscr{O}(U)$ with $F = f$ on D.

Lemma 4.55. *(Fornæss) Let $n > 1$ and let $\Omega \subset \mathbb{C}^n$ be an open set that is not an open set of holomorphy. Then there is a $p \in \Omega$ and $\rho > r > 0$ such that $B_n(p,r) \subset \Omega$, $B_n(p,\rho) \not\subset \Omega$ and every $f \in \mathscr{O}(\Omega)$ extends holomorphically from $B_n(p,r)$ to $B_n(p,\rho)$.*

Proof. Since Ω is not an open set of holomorphy, there are domains D, U with $U \not\subset \Omega$, $\emptyset \neq D \subset U \cap \Omega$ such that every $f \in \mathscr{O}(\Omega)$ extends holomorphically from D to U. Choose $x \in D$ and $y \in U \setminus \Omega$. There is a continuous curve $\gamma(t)$, $0 \leq t \leq 1$, lying in U with $\gamma(0) = x$ and $\gamma(1) = y$. Let β be the least number so that $w := \gamma(\beta) \in \partial\Omega$. Choose a $\rho > 0$ with $B_n(w, 2\rho) \subset\subset U$. Choose $\alpha \in (0, \beta)$ so that $|\gamma(\alpha) - \gamma(\beta)| < \rho$. Let $p = \gamma(\alpha)$. Then $w \in B_n(p,\rho)$. Choose $r > 0$ so that $B_n(p,r) \subset \Omega$. Then every $f \in \mathscr{O}(\Omega)$ extends holomorphically from $B_n(p,r)$ to $B_n(p,\rho)$. $\square$

Theorem 4.56. *Let $\Omega \subset \mathbb{C}^n$ be an open set with $H^q_{\overline{\partial}}(\Omega) = 0$ for $q = 1, \ldots, n-1$. Then Ω is an open set of holomorphy.*

Proof. We use induction on n. When $n = 1$ the statement is true since every open set in $\mathbb{C}$ is an open set of holomorphy.

Now assume $n > 1$ and statement is true for $n - 1$. Suppose, if possible, that $\Omega \subset \mathbb{C}^n$ satisfies the hypothesis but is not an open set of holomorphy. By Lemma 4.55, there is a $p \in \Omega$ and $\rho > r > 0$ such that $B_n(p,r) \subset \Omega$, $B_n(p,\rho) \not\subset \Omega$ and every $g \in \mathscr{O}(\Omega)$ extends holomorphically from $B_n(p,r)$ to $B_n(p,\rho)$. Let $w \in B_n(p,\rho) \cap \partial\Omega$. Change coordinates so that $p = 0$, $w = (0, \ldots, 0, 1)$. Let $D = \Omega \cap \{z_1 = 0\}$. By Theorem 4.54, $H^q_{\overline{\partial}}(D) = 0$ for $q = 1, \ldots, n-2$. By the induction hypothesis, D is an open set of holomorphy. Let $f \in \mathscr{O}(D)$. By Lemma 4.53, there is an $F \in \mathscr{O}(\Omega)$ with $f = F$ on D. Since F extends holomorphically from $B_n(p,r)$ to $B_n(p,\rho)$, it follows that each $f \in \mathscr{O}(D)$ extends holomorphically from $B_{n-1}(p,r)$ to $B_{n-1}(p,\rho)$ with $B_{n-1}(p,\rho) \not\subset D$. This contradicts the assertion that D is n open set of holomorphy. Therefore, Ω is an open set of holomorphy. $\square$

5. Pseudoconvexity

Definition 5.1. Let X be a topological space. A function $u : X \to [-\infty, \infty]$ is said to be *upper semicontinuous* if the set $\{x \in X : u(x) < \alpha\}$ is open for each $\alpha \in \mathbb{R}$.

Exercise 5.2. Let X be a topological space and let $u : X \to [-\infty, \infty]$. Then u is upper semincontinuous if and only if for each $x \in X$, $\overline{\lim}_{y \to x, y \neq x} u(y) \leq u(x)$.

Exercise 5.3. Let K be a compact subset of metric space X and let u be an upper semicontinuous function on K. Then there is a point $a \in K$ such that $u(a) = \sup_{x \in K} u(x)$.

Exercise 5.4. Let $f : \mathbb{R} \to [0, 1]$ be defined by $f(p/q) = 1/q$ for p, q coprime integers and $q > 0$, and $f(x) = 0$ for x irrational. Show that f is upper semicontinuous.

Theorem 5.5. *Let u be an upper semicontinuous function on a metric space (X, d), and suppose that u is bounded above on X. Then there exist continuous functions $\varphi_n : X \to \mathbb{R}$ such that $\varphi_1 \geq \varphi_2 \geq \cdots$ on X and $\lim \varphi_j = u$.*

Proof. We assume that $u \not\equiv -\infty$. For $j \geq 1$ define φ_j by

$$\varphi_j(x) = \sup_{y \in X} (u(y) - jd(x, y)).$$

It is clear that $\varphi_1 \geq \varphi_2 \geq \cdots \geq u$. Since

$$\varphi_j(x) \geq \sup_{y \in X} (u(y) - jd(x_1, y) - jd(x, x_1)) = \varphi_j(x_1) - jd(x, x_1),$$

it follows that $|\varphi_j(x) - \varphi_j(x_1)| \leq jd(x, x_1)$ and φ_j is continuous. For $r > 0$ let $D(x, r) = \{y \in X : d(x, y) < r\}$. Then

$$\begin{aligned}\varphi_j(x) &= \max(\sup_{y \in D(x,r)} (u(y) - jd(x, y)), \sup_{y \notin D(x,r)} (u(y) - jd(x, y))) \\ &\leq \max(\sup_{y \in D(x,r)} u(y), \sup_{y \in X} (u(y) - jr)).\end{aligned}$$

It follows that

$$u(x) \leq \lim_{j \to \infty} \varphi_j(x) \leq \sup_{y \in D(x,r)} u(y).$$

Letting $r \to 0$ yields that $\lim_{j \to 0} \varphi_j(x) = u(x)$, since u is upper semicontinuous. □

Exercise 5.6. Let E be a G_δ set in a topological space X. Then there is an upper semicontinuous function f on X such that $0 \leq f \leq 1$, $E = \{f = 0\}$, and $E \subset S(f)$, where $S(f)$ is the set of points at which f is continuous. In addition, if E is dense, then $E = S(f)$.

Hint: Let $E = \cap E_n$, where $\{E_j\}$ is a descending sequence of open sets with $E_1 = X$. Let $f = (\sup_n n\chi_{E_n})^{-1}$. Then

$$\{f < \alpha\} = \begin{cases} \emptyset & \alpha \leq 0 \\ E_{\lfloor 1/\alpha \rfloor + 1} & 0 < \alpha \leq 1 \\ \mathbb{R} & 1 < \alpha \end{cases}$$

and $\{f < \alpha\}$ is open. If $x_0 \in E$, then $f(x_0) = 0 \leq \liminf f(x) \leq \limsup f(x) \leq f(x_0)$, and hence $x_0 \in S(f)$. Suppose E is dense and $x_0 \in E^c$. Then $\liminf f(x) = 0 < f(x_0)$, and hence $x_0 \notin S(f)$.

Definition 5.7. Let U be an open subset of $\mathbb{C}$. A function $u : U \to [-\infty, \infty)$ is said to be *subharmonic* if it is upper semicontinuous and satisfies the *local sub-mean inequality*, i.e., given $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})\, dt, \quad 0 \leq r < \rho.$$

Exercise 5.8. Let $u : U \to [-\infty, \infty)$ be a subharmonic function and let $x \in U$. Prove that

$$u(x) = \limsup_{y \to x, y \neq x} u(y) = \lim_{\delta \to 0} (\sup\{u(y) : 0 < |y - x| < \delta\}).$$

Exercise 5.9. $\log|z - a|$ and $\log^+|z - a|$ are subharmonic. If $f \in \mathscr{O}(\Omega)$ then $\log|f(z)|$ is subharmonic in Ω.

Exercise 5.10. Let u, v be subharmonic functions in U. Then
(a) $\max(u, v)$ is subharmonic in U;
(b) $\alpha u + \beta v$ is subharmonic in U for all $\alpha, \beta \geq 0$.

Exercise 5.11. Let φ be a convex increasing function on $\mathbb{R}$ and set $\varphi(-\infty) = \lim_{x \to -\infty} \varphi(x)$. Then $\varphi \circ u$ is subharmonic if u is subharmonic.

Exercise 5.12. If $f \in \mathscr{O}(\Omega)$ then $|f(z)|^\alpha$ is subharmonic in Ω.

Theorem 5.13. *(Maximum Principle) Let u be a subharmonic function in a domain $\Omega \subset \mathbb{C}$. If for some $a \in \Omega$, $u(a) = \sup_{z \in \Omega} u(z)$, then u is constant.*

Proof. Let $M = \sup_{z \in \Omega} u(z)$ and $Q = \{z \in \Omega : u(z) = M\}$. Then Q is non-empty because $a \in Q$, and Q is relatively closed since u is upper semicontinuous. The proof will be complete if we show that Q is open. Let $b \in Q$. Suppose ρ is the number such that the sub-mean inequality at b holds for $r < \rho$. Seeking for a contradiction suppose that there is a point $w := b + re^{i\theta}$ with $r < \rho$ and $u(w) < M$. Then $M = u(b) \leq (2\pi)^{-1} \int_0^{2\pi} u(b + re^{it})\, dt < M$, since $u \leq M$ on the circle and $u < M$ in a neighborhood of w. The contradiction shows that $\Delta(b, \rho) \subset Q$ and Q is open. $\square$

Let Q be a bounded subset of an open set $\Omega \subset \mathbb{R}^N$. Then the following are equivalent. (This is similar to Exercise 4.14.)
(i) $\overline{Q} \cap \Omega$ is compact.
(ii) $\overline{Q} \subset \Omega$.
(iii) $d(Q, \partial\Omega) > 0$.
In this case we say Q is relatively compact in Ω and we write $Q \subset\subset \Omega$.

Theorem 5.14. *Let u be defined in a domain $\Omega \subset \mathbb{C}$ with values in $[-\infty, \infty)$ and assume that u is upper semicontinuous.Then the following are equivalent.*
(i) u is subharmonic.
(ii) For every compact set $K \subset \Omega$ and every continuous function h on K which is harmonic in the interior of K and is $\geq u$ on the boundary of K we have $u \leq h$ on K.
(iii) For every disc $D := \Delta(w, r) \subset\subset \Omega$ and every continuous function h on $\overline{D}$ which is harmonic in D and is $\geq u$ on ∂D we have $u \leq h$ on D.
(iv) If $D := \Delta(w, r) \subset\subset \Omega$ and f is a holomorphic polynomial such that $u \leq \Re f$ on ∂D, it follows that $u \leq f$ in D.

(v) If $D := \Delta(w,r) \subset\subset \Omega$ and g is a continuous function on ∂D such that $u \le g$ on ∂D, it follows that $u(w) \le (2\pi)^{-1}\int_0^{2\pi} g(w+re^{it})\,dt$.

(vi) u satisfies the global sub-mean inequality: if $D := \Delta(w,r) \subset\subset \Omega$ then $u(w) \le (2\pi)^{-1}\int_0^{2\pi} u(w+re^{it})\,dt$.

Proof. (i) ⇒ (ii): Let U be a connected component of $\operatorname{int} K$. By the maximum principle, $\max_{\overline{U}}(u-h) = \max_{\partial U}(u-h) \le 0$. Thus $u \le h$ on $\overline{U}$.

(ii) ⇒ (iii) ⇒ (iv) and (vi) ⇒ (i) : Trivial.

(iv) ⇒ (v): Let $\varepsilon > 0$. There is a trigonometric polynomial $\varphi(e^{it}) := \Re\sum_{j=1}^N a_j e^{ijt}$ such that $\varphi(e^{it}) < g(w+re^{it}) < \varphi(e^{it})+\varepsilon$ for all t. Let $f(z) = \sum_{j=0}^N b_j(z-w)^j$, where $b_j = a_j/r^j$. Then $\varphi(e^{it}) = \Re f(w+re^{it})$ for all t, and hence $u < \Re f + \varepsilon$ on ∂D. Thus $u < \Re f + \varepsilon$ on $\overline{D}$. It follows that

$$u(w) < \Re f(w) + \varepsilon = \frac{1}{2\pi}\int_0^{2\pi} \Re f(w+re^{it})\,dt + \varepsilon < \frac{1}{2\pi}\int_0^{2\pi} g(w+re^{it})\,dt + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired inequality.

(v) ⇒ (vi): By Theorem 5.5, there is a sequence $\{u_j\}$ of continuous functions on ∂D such that $u_j \searrow u$. Thus $u(w) \le (2\pi)^{-1}\int_0^{2\pi} u_j(w+re^{it})\,dt$ for all j. By the monotone convergence theorem, $u(w) \le (2\pi)^{-1}\int_0^{2\pi} u(w+re^{it})\,dt$. □

Exercise 5.15. If u is a subharmonic function in Ω and $\overline{\Delta}(z,r) \subset \Omega$, then for $0 \le \rho < r$ and $0 \le \theta < 2\pi$

$$u(z+\rho e^{i\theta}) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2 - 2r\rho\cos(t-\theta)+\rho^2} u(z+re^{it})\,dt.$$

Exercise 5.16. If u is a subharmonic function in Ω and $\overline{\Delta}(z,r) \subset \Omega$, then for $0 \le s < r$,

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+se^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})\,dt.$$

Hint: For $0 \le \rho < r$, let

$$h(\rho e^{i\theta}) := \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2 - 2r\rho\cos(t-\theta)+\rho^2} u(z+re^{it})\,dt.$$

The kernel equals $2\Re[(\zeta/(\zeta-\xi)] - 1$, where $\zeta = re^{it}$, $\xi = \rho e^{i\theta}$. Then h is harmonic in $\Delta(0,r)$ and $h(w) \ge u(z+w)$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+se^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} h(se^{it})\,dt = h(0) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})\,dt.$$

Exercise 5.17. If u, v are functions on Ω such that $\log u$, $\log v$ are subharmonic then $\log(u+v)$ is subharmonic.

Hint: Suppose that $D := \Delta(w,r) \subset\subset \Omega$ and that f is a holomorphic polynomial with $\Re f \ge \log(u+v)$ on ∂D. Then $(u+v)|e^{-f}|$ is subharmonic and ≤ 1 on ∂D, hence it is ≤ 1 in D.

Exercise 5.18. Let $\{u_\alpha\}$ be a family of subharmonic functions in Ω and $u(z) := \sup_\alpha u_\alpha(z)$ for $z \in \Omega$. If $u < \infty$ and u is upper semicontinuous then u is subharmonic.

Exercise 5.19. Let $\{u_j\}$ be a sequence of subharmonic functions in Ω with $u_j \geq u_{j+1}$ for all j. Then $\lim u_j$ is subharmonic.

Example 5.20. The function $u(z) := \sum_j 2^{-j} \log |z - 2^{-j}|$ is subharmonic, and it is discontinuous at 0. The function $v := e^u$ is subharmonic and locally bounded, and it is discontinuous at 0.

Example 5.21. Let $\Omega \subset \mathbb{C}$ be an open set and let $u(z) = -\log d(z, \partial\Omega)$ for $z \in \Omega$. Then u is subharmonic in Ω.

Proof. $u(z) = \sup_{w \in \partial\Omega}(-\log |z - w|)$. □

Theorem 5.22. *If $u \in C^2(\Omega)$ then u is subharmonic if and only if $\Delta u \geq 0$.*

Proof. Suppose that $\Delta u \geq 0$. Let $u_\varepsilon(z) = u(z) + \varepsilon |z|^2$. Suppose that $D := \Delta(w, r) \subset\subset \Omega$ and that f is a holomorphic polynomial with $u_\varepsilon - \Re f \leq 0$ on ∂D. The function $v_\varepsilon := u_\varepsilon - \Re f$ satisfies $\Delta v_\varepsilon \geq 4\varepsilon$, so it cannot reach a local maximum at any point of D. It follows that $v_\varepsilon \leq 0$ in D. Therefore u_ε is subharmonic for $\varepsilon > 0$. By Exercise 5.19, u is subharmonic.

Conversely, suppose that u is subharmonic. We show that $\Delta u \geq 0$. Suppose, if possible, that $\Delta u(w) < 0$ for some $w \in \Omega$. Then $\Delta u < 0$ on $D := \Delta(w, r)$ for some $r > 0$. By what has been proved, $-u$ is subharmonic on D. It follows that u is harmonic on D, contradicting the assumption that $\Delta u(w) < 0$. Therefore, $\Delta u \geq 0$. □

Exercise 5.23. Let u be a subharmonic function in a domain $\Omega \subset \mathbb{C}$ that is not identically $-\infty$. Show that u is locally integrable.

Let $\psi \in C_0^\infty(\mathbb{C})$ be a radial function (i.e., $\psi(z) = \psi(|z|)$) with $\operatorname{supp} \psi \subset \Delta(0, 1)$ and $\int_{\mathbb{C}} \psi(z)\, dm(z) = 1$. Let $\psi_j(z) = j^2 \psi(jz)$. Then $\int \psi_j \, dm = 1$.

For $z \in \Omega$ let $\delta_\Omega(z) = d(z, \partial\Omega)$.

Theorem 5.24. *Let u be a subharmonic function in a domain $\Omega \subset \mathbb{C}$. Let*

$$u_j(z) = \int_{|\zeta| < 1/j} u(z + \zeta) \psi_j(\zeta)\, dm(\zeta)$$

for $z \in \Omega_j := \{w \in \Omega : \delta_\Omega(w) > 1/j\}$. Then u_j is a C^∞ subharmonic function on Ω_j, $u_j \geq u_{j+1}$ for all j, and $\lim u_j = u$.

Proof. The function u_j is C^∞ because $u_j(z) = \int_{|\zeta|<1/j} u(\zeta) \psi_j(\zeta - z)\, dm(\zeta)$. Since

$$\begin{aligned} u_j(z) &= \int_{|\zeta|<1} u(z + \zeta/j) \psi(\zeta)\, dm(\zeta) \\ &= \int_0^1 \Big(\int_0^{2\pi} u(z + re^{it}/j)\, dt \Big) \psi(r) r \, dr, \end{aligned}$$

it follows from Exercise 5.16 and the sub-mean inequality that $u_j \geq u_{j+1} \geq u$. By Fatou lemma, we have

$$u(z) \leq \lim u_j(z) \leq \int_{|\zeta|<1} \overline{\lim_{j \to \infty}}\, u(z + \zeta/j) \psi(\zeta)\, dm(\zeta) \leq u(z).$$

□

Theorem 5.25. *Let u be a subharmonic function in Ω and let $h \in \mathscr{O}(D)$ with $f(D) \subset \Omega$. Then $u \circ h$ is subharmonic in D.*

Proof. Let $\{u_j\}$ be a sequence of C^∞ subharmonic functions with $u_j \searrow u$. Then $u_j \circ h$ is subharmonic by Theorem 5.22. Thus $u \circ h$ is subharmonic by Exercise 5.19. □

Definition 5.26. Let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function on an open set $\Omega \subset \mathbb{C}^n$. The function u is said to be *plurisubharmonic* if for $a \in \Omega, b \in \mathbb{C}^n$, the function $u_{a,b} := u(a + b\zeta)$ is subharmonic in the open set $\{\zeta \in \mathbb{C} : a + b\zeta \in \Omega\}$. The set of plurisubharmonic functions in Ω is denoted by $\mathscr{P}(\Omega)$.

Proposition 5.27. *Let $u \in C^2(\Omega)$. Then u is plurisubharmonic in Ω if and only if for all $z \in \Omega$, $w \in \mathbb{C}^n$,*

$$L_u(z, w) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0.$$

Proof. Exercise. □

Exercise 5.28. Suppose that $u \in \mathscr{P}(\Omega)$ and $\overline{B}_n(w, r) \subset \Omega$ and $0 < s < r$. Then

$$u(w) \leq \sigma_n^{-1} \int_{\partial B_n} u(w + s\zeta)\, d\sigma(\zeta) \leq \sigma_n^{-1} \int_{\partial B_n} u(w + r\zeta)\, d\sigma(\zeta).$$

Exercise 5.29. Let $u \in \mathscr{P}(\Omega)$ and suppose that u is not identically $-\infty$ in any connected component of Ω. Then u is locally integrable.

Let $\psi \in C_0^\infty(\mathbb{C}^n)$ be a radial function such that $\int \psi\, dV = 1$ and $\operatorname{supp} \psi \subset B_n$. Let $\psi_j(z) = j^{2n} \psi(jz)$.

Proposition 5.30. *Suppose that u is a plurisubharmonic function in an open set $\Omega \subset \mathbb{C}^n$. Then*

$$u_j(z) := \int u(z + \zeta) \psi_j(\zeta)\, dV(\zeta)$$

is a C^∞ plurisubharmonic function in the open set $\Omega_j := \{z \in \Omega : \delta_\Omega(z) > 1/j\}$, and the sequence $\{u_j\}$ decreases and converges to u.

Theorem 5.31. *Let h be a holomorphic mapping from $\Omega \subset \mathbb{C}^n$ to $D \subset \mathbb{C}^m$. If $u \in \mathscr{P}(D)$, then $u \circ h \in \mathscr{P}(\Omega)$.*

Proof. It is true for C^2 functions. Then use approximation. □

Corollary 5.32. *If $f \in \mathscr{O}(\Omega)$ then $\log |f| \in \mathscr{P}(\Omega)$.*

Definition 5.33. Let K be a compact subset of an open set $\Omega \subset \mathbb{C}^n$. The *plurisubharmonic hull* of K in Ω is defined by

$$\hat{K}_{\mathscr{P}(\Omega)} := \{z \in \Omega : u(z) \leq \max_{w \in K} u(w) \text{ for each } u \in \mathscr{P}(\Omega)\}.$$

Exercise 5.34. Show that $\hat{K}_{\mathscr{P}(\Omega)} \subset \hat{K}_{\mathscr{O}(\Omega)}$.

Definition 5.35. An open set $\Omega \subset \mathbb{C}^n$ is said to be *pseudoconvex* if for each compact $K \subset \Omega$, $\hat{K}_{\mathscr{P}(\Omega)}$ is relatively compact in Ω.

Remark. It is defined this way because at this point it is not clear whether $\hat{K}_{\mathscr{P}(\Omega)}$ is relatively closed. In the definition of holomorphic convexity, $\hat{K}_{\mathscr{O}(\Omega)}$ is required to be compact. Note that "ralatively compact" does not imply "relatively closed". "Relatively compact" and "relatively closed" together imply "compact". Later we will see that in case Ω is pseudoconvex $\hat{K}_{\mathscr{P}(\Omega)}$ is relatively closed and therefore compact.

Exercise 5.36. An open set Ω is pseudoconvex if and only if each connected component of Ω is pseudoconvex.

Theorem 5.37. *If Ω is holomorphically convex, then Ω is pseudoconvex. In particular, a convex domain is pseudoconvex.*

Proof. This is a consequence of Exercise 5.34. □

5.38. Levi Problem.

The problem of geometric characterization of domains of holomorphy is called the Levi problem. Specifically, the Levi problem is to show that the converse to the above is true, i.e., each pseudoconvex domain is a domain of holomorphy. It was posed by E.E. Levi in 1911. The Levi problem was first solved by K. Oka in 1942 in $\mathbb{C}^2$, and, in arbitrary dimension, it was solved independently by Oka, H. Bremermann, and F. Norguet in the early 1950's. We will present Hörmander's proof in next chapter.

Definition 5.39. Let Ω be an open set in $\mathbb{C}^n$ and let $u : \Omega \to [-\infty, \infty)$. The function u is said to be an *exhaustion function* for Ω if $\{u \leq c\} \subset\subset \Omega$ for each $c \in \mathbb{R}$.

Exercise 5.40. Show that $|z|^2$ is a plurisubharmonic exhaustion function for $\mathbb{C}^n$ and that $1/(1-|z|^2)$ is a plurisubharmonic exhaustion function for B_n. Find a plurisubharmonic exhaustion function for $\Delta^n(0,1)$.

Proposition 5.41. *If $\Omega \subset \mathbb{C}^n$ has a plurisubharmonic exhaustion function, then Ω is pseudoconvex.*

Proof. Let $K \subset \Omega$ be compact and let u be a plurisubharmonic exhaustion function for Ω. Then $c := \sup_{w \in K} u(w) \in \mathbb{R}$ and hence $\hat{K}_{\mathscr{P}(\Omega)} \subset \{w \in \Omega : u(w) \leq c\} \subset\subset \Omega$. □

Let $\Delta := \Delta(0,1)$. If $\varphi : \partial\Delta \to \mathbb{C}^n$ is a continuous map which is holomorphic in Δ, the set $\varphi(\overline{\Delta})$ is called an *analytic disc* in $\mathbb{C}^n$. If $S := \varphi(\overline{\Delta})$ is an analytic disc, we shall call $\partial S := \varphi(\partial\Delta)$ the *boundary* of S. Note that this is different from the boundary of S as a set.

Definition 5.42. Let Ω be an open set in $\mathbb{C}^n$. The set Ω is said to satisfy the *continuity principle* if whenever $\{S_\alpha\}$ is a family of analytic discs in Ω with $\cup\partial S_\alpha \subset\subset \Omega$ one has $\cup S_\alpha \subset\subset \Omega$.

Remark. This is in analogue with Exercise 4.5 (iii).

Proposition 5.43. *If Ω is a pseudoconvex open set in $\mathbb{C}^n$ then Ω satisfies the continuity principle.*

Proof. Suppose that $Q := \cup\partial S_\alpha \subset\subset \Omega$ and let $K = \overline{Q}$. Then K is a compact subset of Ω. By the maximum principle, $\cup S_\alpha \subset \hat{K}_{\mathscr{P}(\Omega)}$. Since Ω is pseudoconvex and therefore $\hat{K}_{\mathscr{P}(\Omega)} \subset\subset \Omega$, it follows that $\cup S_\alpha \subset\subset \Omega$. □

Proposition 5.44. *Let Ω be an open set in $\mathbb{C}^n$. Then Ω satisfies the continuity principle if and only if for every analytic disc S in Ω, $d(S, \partial\Omega) = d(\partial S, \partial\Omega)$.*

Remark. This is similar to Exercise 4.5 (iv).

Proof. $\Leftarrow$ is clear. Assume that Ω satisfies the continuity principle. Suppose, if possible, that there is an analytic disc $S \subset \Omega$ with $d(S, \partial\Omega) < d(\partial S, \partial\Omega)$. Choose $p \in S$ and $q \in \partial\Omega$ so that $|q - p| = d(S, \partial\Omega)$. For $0 \leq t < 1$ let $S_t = S + t(q - p) := \{z + t(q - p) : z \in S\}$. Then the family $\{S_t : 0 \leq t < 1\}$ violates the continuiuty principle. □

The basic Hartogs figure in $\mathbb{C}^n$ (with $n \geq 2$) is the couple (Λ, Λ_0), where

$$\Lambda_0 = \{(z_1, 0, \ldots, 0) \in \mathbb{C}^n : |z_1| \leq 1\} \cup \{(z_1, z_2, 0, \ldots, 0) : |z_1| = 1, |z_2| \leq 1\}$$
$$\Lambda = \{(z_1, z_2, 0, \ldots, 0) : |z_1| \leq 1, |z_2| \leq 1\}.$$

A couple (Γ, Γ_0) is said to be a Hartogs figure if there is an injective holomorphic mapping f from a neighborhood of Λ to $\mathbb{C}^n$ such that $f(\Lambda) = \Gamma$ and $f(\Lambda_0) = \Gamma_0$.

Definition 5.45. An open set Ω is said to be *Hartogs pseudoconvex* if whenever (Γ, Γ_0) is a Hartogs figure with $\Gamma_0 \subset \Omega$ we have $\Gamma \subset \Omega$.

Recall that $\Delta := \Delta(0, 1)$.

Proposition 5.46. *Let Ω be an open set in $\mathbb{C}^n$. If Ω satisfies the continuity principle, then Ω is Hartogs pseudoconvex.*

Proof. Suppose that (Γ, Γ_0) is a Hartogs figure with $\Gamma_0 \subset \Omega$. Let f be an injective holomorphic map from a neighborhood of Λ to $\mathbb{C}^n$ with $f(\Lambda) = \Gamma$ and $f(\Lambda_0) = \Gamma_0$. For $w \in \Delta$ let $S_w = \{f(z_1, w, 0, \ldots, 0) \in \mathbb{C}^n : |z_1| \leq 1\}$. Let $D = \{w \in \Delta : S_w \subset \Omega\}$. Then D is an open set containing 0 since Ω is open. The family $\{S_w : w \in D\}$ satisfies $\cup_{w \in D} \partial S_w \subset \Gamma_0 \subset\subset \Omega$. By the continuity principle, $Q := \cup_{w \in D} S_w \subset\subset \Omega$, which means that $\overline{Q} \subset \Omega$. This in turn implies that D is relatively closed in Δ. Since Δ is connected we see that $D = \Delta$ and $\overline{Q} = \Gamma$. Therefore $\Gamma \subset \Omega$. □

Let Ω be an open set in $\mathbb{C}^n$. For $u \in \mathbb{C}^n$, $|u| = 1$, let

$$\delta_u(z) = \sup\{\tau > 0 : z + \tau\zeta u \in \Omega \text{ for } \zeta \in \Delta\}.$$

The number $\delta_u(z)$ measures how large a disc in the u-direction with center z is contained in Ω. Let $\delta_\Omega(z) = d(z, \partial\Omega)$. Then

$$\delta_\Omega(z) = \inf\{\delta_u(z) : u \in \mathbb{C}^n \text{ with } |u| = 1\}.$$

Theorem 5.47. *Let Ω be an open set in $\mathbb{C}^n$. If Ω is Hartogs pseudoconvex, then $-\log \delta_\Omega$ is plurisubharmonic.*

Proof. Since $-\log \delta_\Omega = \sup_u(-\log \delta_u)$, it suffices to show that for any unit vectors u, w and any $a \in \Omega$, the function $h(\lambda) := -\log \delta_u(a + \lambda w)$ is subharmonic in the open set $\Omega_{a,w} := \{\lambda \in \mathbb{C} : a + \lambda w \in \Omega\}$.

Case 1. u, w are linearly dependent. Then there is a $c \in \mathbb{C}$ with $|c| = 1$ such that $u = cw$. In this case,

$$\begin{aligned}\delta_u(a + \lambda w) &= \sup\{\tau > 0 : a + \lambda w + \tau\zeta \cdot cw \in \Omega \text{ for } \zeta \in \Delta\} \\ &= \sup\{\tau > 0 : a + (\lambda + \tau c\zeta)w \in \Omega \text{ for } \zeta \in \Delta\} \\ &= \sup\{\tau > 0 : \lambda + \tau c\zeta \in \Omega_{a,w} \text{ for } \zeta \in \Delta\} \\ &= \delta_{\Omega_{a,w}}(\lambda).\end{aligned}$$

Hence $h(\lambda) = -\log \delta_{\Omega_{a,w}}(\lambda)$, which is suharmonic by Example 5.21.

Case 2. u, w are linearly independent. It suffices to show that if $\overline{\Delta}(p, r) \subset \Omega_{a,w}$ and if g is a holomorphic polynomial with

$$h(p + r\zeta) = -\log \delta_u(a + (p + r\zeta)w) \leq \Re g(\zeta) \tag{12}$$

for $|\zeta| = 1$, then (12) also holds for $|\zeta| \leq 1$. Now (12) is equivalent to

$$\delta_u(b + r\zeta w) \geq |e^{-g(\zeta)}|,$$

or

$$b + r\zeta w + \eta e^{-g(\zeta)} u \in \Omega \text{ for } \eta \in \Delta, \tag{13}$$

where $b := a + pw$. Therefore it suffices to show that if (13) holds for $|\zeta| = 1$ then it holds for $|\zeta| \leq 1$.

For $0 < \tau < 1$, let

$$\begin{aligned}\Gamma_{\tau,0} &= \{b + r\zeta w + \tau\eta e^{-g(\zeta)}u : |\zeta| = 1, |\eta| \leq 1\} \cup \{b + r\zeta w : |\zeta| \leq 1\}, \\ \Gamma_\tau &= \{b + r\zeta w + \tau\eta e^{-g(\zeta)}u : |\zeta| \leq 1, |\eta| \leq 1\}.\end{aligned}$$

Choose $u_3, \dots, u_n \in \mathbb{C}^n$ so that $w, u, u_3, \dots, u_n$ form a basis for $\mathbb{C}^n$. Define $f : \mathbb{C}^n \to \mathbb{C}^n$ by

$$f(z_1, \dots, z_n) = b + rz_1 w + \tau z_2 e^{-g(z_1)}u + z_3 u_3 + \cdots + z_n u_n.$$

Then f is an injective holomorphic mapping with $f(\Lambda) = \Gamma_\tau$ and $f(\Lambda_0) = \Gamma_{\tau,0}$. Thus $(\Gamma_\tau, \Gamma_{\tau,0})$ is a Hartogs figure. Assume that (12) holds, and hence (13) holds, for $|\zeta| = 1$. Then $\Gamma_{\tau,0} \subset \Omega$ for every τ with $0 < \tau < 1$. Since Ω is Hartogs pseudoconvex, it follows that $\Gamma_\tau \subset \Omega$ for every τ with $0 < \tau < 1$. This means that (13) holds, and hence (12) holds, for $|\zeta| \leq 1$. Therefore $h(\lambda)$ is plurisubharmonic in $\Omega_{a,w}$. □

Definition 5.48. A function $u \in \mathscr{P}(\Omega)$ is said to be *strictly plurisubharmonic* if for each $a \in \Omega$ there is a neighborhood U of a and an $\varepsilon > 0$ such that the function $u(z) - \varepsilon|z|^2$ is plurisubharmonic in U.

Proposition 5.49. *Let $u \in C^2(\Omega)$. Then u is strictly plurisubharmonic if and only if*

$$L_u(z, w) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k > 0$$

for $z \in \Omega$ and $0 \neq w \in \mathbb{C}^n$.

Proof. Exercise. □

Let $\psi \in C_0^\infty(\mathbb{C}^n)$ be a radial function such that $\int \psi\, dV = 1$ and $\operatorname{supp} \psi \subset B_n$. Let $\psi_j(z) = j^{2n}\psi(jz)$.

Proposition 5.50. *Suppose that u is a plurisubharmonic function in an open set $\Omega \subset \mathbb{C}^n$. Then*

$$u_j(z) := \int u(z+\zeta)\psi_j(\zeta)\, dV(\zeta) + (1/j)|z|^2$$

is a C^∞ strictly plurisubharmonic function in the open set $\Omega_j := \{z \in \Omega : \delta_\Omega(z) > 1/j\}$, and the sequence $\{u_j\}$ decreases and converges to u. If in addition u is continuous then $\{u_j\}$ converges to u uniformly on each compact subset of Ω.

Proof. This is a consequence of Proposition 5.30. When u is continuous and $K \subset \Omega$ is compact, $u_j \to u$ uniformly on K by Dini theorem. □

Exercise 5.51. Let $\chi(t) = 0$ for $t \le 0$ and $\chi(t) = \exp(t - 2t^{-1})$ for $t > 0$. Show that $\chi \in C^\infty(\mathbb{R})$ and $\chi', \chi'' > 0$ for $t > 0$.

Theorem 5.52. *Let u be a continuous plurisubharmonic exhaustion function for an open set Ω in $\mathbb{C}^n$, and let K be a compact subset of Ω. Then for each $\varepsilon > 0$ there is a C^∞ strictly plurisubharmonic exhaustion function φ for Ω with $\varphi > u$ on Ω and $u < \varphi < u + \varepsilon$ on K.*

Proof. Let $\Omega_j = \{z \in \Omega : u(z) < j\}$ for $j = 0, 1, 2, \ldots$. Then

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega,$$

and by adding a suitable constant to u we may assume that $K \subset \Omega_0$. For $\varepsilon > 0$ let $D_\varepsilon = \{z \in \Omega : \delta_\Omega(z) > \varepsilon\}$. Choose $\{\varepsilon_j\}$ so that $\Omega_{j+2} \subset\subset D_{\varepsilon_j}$ for $j = 0, 1, 2, \ldots$. By Proposition 5.50 there are C^∞ strictly plurisubharmonic functions v_j on D_{ε_j} such that $u < v_0 < u + \varepsilon$ on $\overline{\Omega}_1$, and $u < v_j < u + 1$ on $\overline{\Omega}_{j+1}$ for $j \ge 1$. Choose $\psi_j \in C_0^\infty(D_{\varepsilon_j})$ so that $\psi_j = 1$ on a neighborhood of $\overline{\Omega}_{j+2}$. Let $u_j = v_j\psi_j$. Then $u_j \in C_0^\infty(\Omega)$ and u_j is strictly plurisubharmonic on Ω_{j+2}, $u < u_0 < u + \varepsilon$ on $\overline{\Omega}_1$, and $u < u_j < u + 1$ on $\overline{\Omega}_{j+1}$ for $j \ge 1$. It follows that $u_j < j$ on Ω_{j-1} and $u_j > j$ on $\overline{\Omega}_{j+1} \setminus \Omega_j$ for $j \ge 1$. By Exercise 5.51 there is a $\chi \in C^\infty(\mathbb{R})$ so that $\chi(t) = 0$ for $t \le 0$ and $\chi(t), \chi'(t), \chi''(t) > 0$ for $t > 0$. For $j \ge 1$, the function $\chi \circ (u_j - j)$ equals 0 on Ω_{j-1}, ≥ 0 and plurisubharmonic on Ω_{j+2}, and > 0 and strictly plurisubharmonic on $\overline{\Omega}_{j+1} \setminus \Omega_j$, since χ is increasing and convex, and since

$$\sum_{k,\ell=1}^n \frac{\partial^2(\chi(u_j(z) - j))}{\partial z_k \partial \overline{z}_\ell} w_k \overline{w}_\ell = \chi'(u_j(z) - j) \sum_{k,\ell=1}^n \frac{\partial^2 u_j(z)}{\partial z_k \partial \overline{z}_\ell} w_k \overline{w}_\ell + \chi''(u_j(z) - j) \Big| \sum_{k=1}^n \frac{\partial u_j(z)}{\partial z_k} w_k \Big|^2.$$

Therefore, we can successively choose integers $m_j \in \mathbb{N}$ so that the function

$$\varphi_k := u_0 + \sum_{j=1}^k m_j \chi \circ (u_j - j)$$

$> u$ and strictly plurisubharmonic on Ω_{k+1}. It is clear that $\varphi_k = u_0$ on Ω_0, and $\varphi_{k+1} = \varphi_k$ on Ω_k. Thus $\varphi := \lim \varphi_k$ has all the required properties. □

Theorem 5.53. *Let $\Omega \subset \mathbb{C}^n$ be an open set. Then the following are equivalent.*
(i) Ω is pseudoconvex.
(ii) Ω satisfies the continuity principle.
(iii) Each analytic disc $S \subset \Omega$ satisfies $d(S, \partial\Omega) = d(\partial S, \partial\Omega)$.
(iv) Ω is Hartogs pseudoconvex.
(v) $-\log \delta_\Omega$ is plurisubharmonic.
(vi) Each compact subset K of Ω satisfies $d(K, \partial\Omega) = d(\hat{K}_{\mathscr{P}(\Omega)}, \partial\Omega)$.
(vii) Ω has a C^∞ strictly plurisubharmonic exhaustion function.
(viii) Ω has a plurisubharmonic exhaustion function.

Proof. (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (i): proved.
(vi) $\Rightarrow$ (i): trivial.
(v) $\Rightarrow$ (vi): Suppose that $z \in \hat{K}_{\mathscr{P}(\Omega)}$. Then $u(z) \le \max_K u$ for each $u \in \mathscr{P}(\Omega)$. In particular, $-\log \delta_\Omega(z) \le \max_{w\in K}(-\log \delta_\Omega(w))$. Thus $\delta_\Omega(z) \ge d(K, \partial\Omega)$. It follows that $d(K, \partial\Omega) = d(\hat{K}_{\mathscr{P}(\Omega)}, \partial\Omega)$. ☐

Theorem 5.54. *Let K be a compact subset of a pseudoconvex open set $\Omega \in \mathbb{C}^n$ and $\zeta \in \Omega \setminus \hat{K}_{\mathscr{P}(\Omega)}$. Then there is a C^∞ strictly plurisubharmonic exhaustion function h for Ω with $h(\zeta) > 0$ and $h < 0$ on K.*

Proof. Let u be a continuous plurisubharmonic exhaustion function for Ω with $u < 0$ on $\{\zeta\} \cup K$. Let $\Omega_k = \{u < k\}$. There is a $v \in \mathscr{P}(\Omega)$ with $v(\zeta) > 1$ and $v < 0$ on K. There is a sequence $\{v_j\}$ of C^∞ plurisubharmonic functions on Ω_3 such that $v_j \searrow v$. Since $K \subset \cup_j\{z \in \Omega_3 : v_j(z) < 0\}$, and since K is compact, there is an $\ell \in \mathbb{N}$ so that $v_\ell < 0$ on K. Let $M = \max_{\overline{\Omega}_2} v_\ell$ and

$$\varphi(z) = \begin{cases} \max(v_\ell(z), Mu(z)) & \text{for } z \in \Omega_2 \\ Mu(z) & \text{for } z \in \Omega \setminus \overline{\Omega}_1. \end{cases}$$

Then φ is a continuous exhaustion function for Ω with $\varphi(\zeta) > 1$ and $\varphi < 0$ on K. By Theorem 5.52, there is a C^∞ strictly plurisubharmonic exhaustion function h for Ω with $\varphi - 1 < h < \varphi$ on the compact set $\{\zeta\} \cup K$. Thus $h(\zeta) > 0$ and $h < 0$ on K. ☐

Corollary 5.55. *Let K be a compact subset of a pseudoconvex open set $\Omega \in \mathbb{C}^n$. Then $\hat{K}_{\mathscr{P}(\Omega)} = \hat{K}_{\mathscr{P}(\Omega)\cap C^\infty(\Omega)}$. Hence $\hat{K}_{\mathscr{P}(\Omega)}$ is compact.*

Remark. In fact, $\hat{K}_{\mathscr{P}(\Omega)} = \hat{K}_{\mathscr{O}(\Omega)}$. See Range, Chap. 6, Sec. 1.8.
Remark. In (Fornaess and Stensones, Conterexamples), p. 66, there is an example of $K \subset \Omega$ for which $\hat{K}_{\mathscr{P}(\Omega)}$ is not relatively closed in Ω.

Theorem 5.56. *Let $\{\Omega_\alpha\}$ be a collection of pseudoconvex open sets in $\mathbb{C}^n$ and let Ω be the interior of $\cap\Omega_\alpha$. Then Ω is pseudoconvex.*

Proof. $-\log \delta_\Omega = \sup(-\log \delta_{\Omega_\alpha})$. ☐

Theorem 5.57. *Let $\{\Omega_j\}$ be an ascending sequence of pseudoconvex open sets in $\mathbb{C}^n$. Then the union $\Omega := \cup\Omega_j$ is pseudoconvex.*

Proof. $-\log \delta_{\Omega_j} \searrow -\log \delta_\Omega$. ☐

Theorem 5.58. *An open set $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if each point $\zeta \in \overline{\Omega}$ has a neighborhood U_ζ such that $U_\zeta \cap \Omega$ is pseudoconvex.*

Proof. First assume Ω is bounded. For each $\zeta \in \partial\Omega$, $-\log \delta_\Omega = -\log \delta_{U_\zeta \cap \Omega}$ near ζ. Thus $-\log \delta_\Omega$ is plurisubharmonic near $\partial\Omega$. So there is a neighborhood U of $\partial\Omega$ such that $-\log \delta_\Omega$ is plurisubharmonic in $U \cap \Omega$. Let $M = \max_{\Omega \setminus U}(-\log \delta_\Omega)$. Then $\max(-\log \delta_\Omega, M+1)$ is a plurisubharmonic exhaustion function for Ω. Thus Ω is pseudoconvex.

If Ω is unbounded, then $\Omega \cap B_n(0,k)$ is pseudoconvex for each $k \in \mathbb{N}$. By Theorem 5.57, Ω is pseudoconvex. $\square$