ON CR PERELMAN'S HARNACK ESTIMATES AND ENTROPY FORMULAE FOR THE COUPLED CR YAMABE FLOW

SHU-CHENG CHANG

In this note, we first derive the CR analogue of Perelman's Harnack estimate (0.4) for the positive solution of the CR conjugate heat equation

$$\frac{\partial u}{\partial t}=-4\Delta_b u+4Wu. \tag{0.1}$$

under the CR Yamabe flow. Then we define the CR Perelman's $\mathcal{F}$-functional (0.5) from this Perelman's Harnack quantity and obtain its monotonicity property for the following coupled CR Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t}\theta\left(t\right)=-2W\left(t\right)\theta\left(t\right), \quad \theta\left(0\right)=\mathring{\theta}, \\ \frac{\partial f}{\partial t}=-4\Delta_b f+4\left|\nabla_b f\right|^2-4W. \end{cases} \tag{0.2}$$

Second, we derive the CR analogue of Perelman's Harnack estimate (0.6) for the positive fundamental solution of the CR conjugate heat equation (0.1) under the CR Yamabe flow. Then we define the CR Perelman's $\mathcal{W}$-functional (0.8) and obtain its monotonicity property for the following coupled CR Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t}\theta\left(t\right)=-2W\left(t\right)\theta\left(t\right), \quad \theta\left(0\right)=\mathring{\theta}, \\ \frac{\partial f}{\partial t}=-4\Delta_b f+4\left|\nabla_b f\right|^2-4W+\frac{2}{\tau}, \\ \frac{d\tau}{dt}=-1. \end{cases} \tag{0.3}$$

Now we state the first Perelman's Li-Yau type Harnack estimate for the positive solution of the CR conjugate heat equation under the CR Yamabe flow in a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion.

Key words and phrases. CR heat equation, CR Yamabe soliton, Li-Yau gradient estimate, Perelman Harnack estimate, Conjugate heat equation, Entropy formulae, Spherical structure, Coupled CR Yamabe flow.

Theorem 0.1. *Let $(M, J, \overset{\circ}{\theta})$ be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Let $(M, J, \theta(t))$ be a solution to the CR Yamabe flow on $M \times [0, T]$ and u be the positive solution to the CR conjugate heat equation (0.1) with $u = e^{-f}$ and $f_0(x, 0) = 0$. Then*

$$\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W - \frac{1}{\tau} \leq 0 \tag{0.4}$$

on $M \times [0, T)$ with $\tau = T - t$.

As pointed out by S.-T. Yau that the derivation of the entropy formula resembles Li-Yau gradient estimate for the heat equation. We define the CR Perelman $\mathcal{F}$-functional by

$$\begin{array}{rcl} \mathcal{F}(\theta(t), f(t)) & = & 4 \int_M [(\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W)] e^{-f} d\mu \\ & = & \int_M (2W + |\nabla_b f|^2) e^{-f} d\mu. \end{array} \tag{0.5}$$

with the constraint $\int_M e^{-f} d\mu = 1$.

We derive the following monotonicity property of CR $\mathcal{F}$-functional.

Theorem 0.2. *Let $(M, J, \overset{\circ}{\theta})$ be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Let $(M, \theta(t), f(t))$ be a solution to the coupled system (0.2) on $M \times [0, T)$ with $f_0(x, 0) = 0$. Then*

$$\begin{array}{rcl} \frac{d}{dt} \mathcal{F}(\theta(t), f(t)) & = & 8 \int_M \left| \left(\nabla^H\right)^2 f + \frac{W}{2} L_\theta \right|^2 u d\mu + 2 \int_M W |\nabla_b f|^2 u d\mu \\ & \geq & 0 \end{array}$$

for all $t \in [0, T)$.

Next we state the second CR Perelman's Li-Yau type Harnack estimate for the positive fundamental solution of the CR conjugate heat equation under the CR Yamabe flow.

Theorem 0.3. *Let $(M, J, \overset{\circ}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Let $(M, J, \theta(t))$ be a solution to the CR Yamabe flow on $M \times [0, T]$ and u be the positive fundamental solution to the CR conjugate heat equation (0.1) with $u = \frac{e^{-f}}{(4\pi\tau)^2}$ and $f_0(x, 0) = 0$. Then*

$$\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W + \frac{1}{8\tau}(f - 4) \leq 0. \tag{0.6}$$

That is

$$f_{\tau}+\left|\nabla_{b}f\right|^{2}-2W+\frac{f}{2\tau}\leq 0 \tag{0.7}$$

on $M\times[0,T)$ *with* $\tau=T-t$.

Again we define the CR Perelman $\mathcal{W}$-functional by

(0.8)

$$\begin{aligned}\mathcal{W}(\theta(t),f(t),\tau) &= 4\int_{M}\tau[\Delta_{b}f-\tfrac{3}{4}\left|\nabla_{b}f\right|^{2}+\tfrac{1}{2}W+\tfrac{1}{8\tau}(f-4)](4\pi\tau)^{-2}e^{-f}d\mu\\ &= \int_{M}[\tau(2W+\left|\nabla_{b}f\right|^{2})+\tfrac{1}{2}(f-4)](4\pi\tau)^{-2}e^{-f}d\mu,\end{aligned}$$

with the constraint $\int_{M}(4\pi\tau)^{-2}e^{-f}d\mu=1$. Note that $\mathcal{W}$ is scalar invariant under $\tau\longmapsto c\tau$ and $\theta\longmapsto c\theta$. Moreover, $\mathcal{W}(\theta,f,\tau)=\mathcal{W}(\Phi^{*}\theta,f\circ\Phi,\tau)$ for a contact diffeomorphism $\Phi:M\rightarrow M$.

We will derive the following monotonicity property of CR $\mathcal{W}$-functional.

Theorem 0.4. *Let* $(M,J,\overset{\circ}{\theta})$ *be a closed spherical pseudohermitian* 3*-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Let* $(M,\theta(t),f(t),\tau(t))$ *be a solution to the coupled CR Yamabe flow on* $M\times[0,T)$ *with* $f_{0}(x,0)=0$. *Then*

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{W}(\theta(t),f(t),\tau(t))\\ &=8\tau\int_{M}\left|\left(\nabla^{H}\right)^{2}f+\tfrac{W}{2}L_{\theta}-\tfrac{1}{4\tau}L_{\theta}\right|^{2}ud\mu\\ &\quad+\tau\int_{M}[2W\left|\nabla_{b}f\right|^{2}+\tfrac{\left|\nabla_{b}f\right|^{2}}{\tau}]ud\mu\\ &\geq 0\end{aligned}$$

for all $t\in[0,T)$.

The classification of CR Yamabe soliton is one of the key steps in understanding the singularity formation of the Yamabe flow.

Theorem 0.5. *(i) Any closed spherical* 3*-dimensional steady CR Yamabe breather with nonnegative Tanaka-Webster curvature and vanishing torsion is a closed pseudohermitian* 3*-manifold of zero Tanaka-Webster curvature and vanishing torsion.*

(ii) Any closed spherical 3*-dimensional shrinking CR Yamabe breather with nonnegative Tanaka-Webster curvature and vanishing torsion is a closed pseudohermitian* 3*-manifold of positive constant Tanaka-Webster curvature and vanishing torsion.*

Finally, if we further assume that our solution to the CR Yamabe flow is of Type I. That is

$$|W| \leq \frac{k_0}{T-t},$$

for some constant k_0. Here T is the blow-up time. Thus we have the Perelman's Li-Yau type Harnack estimate for the positive solution of the CR conjugate heat equation under the CR Yamabe flow in a closed spherical pseudohermitian 3-manifold with vanishing torsion.

Theorem 0.6. *Let $(M, J, \mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with vanishing torsion. Let $(M, J, \theta(t))$ be a solution of Type I to the CR Yamabe flow (**??**) on $M \times [0, T)$ and u be the positive solution to the CR conjugate heat equation (0.1) with $u = e^{-f}$ and $f_0(x, 0) = 0$. Then, for some $\lambda_0 = \lambda_0(k_0) > 0$*

$$\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W - \frac{\lambda_0}{\tau} \leq 0$$

on $M \times [0, T)$ with $\tau = T - t$.

Acknowledgments. The first author would like to express his thanks to Prof. S.-T. Yau for his inspirations from Li-Yau gradient estimate, Prof. C.-S. Lin, director of Taida Institute for Mathematical Sciences, NTU, for constant encouragement and supports. The work is not possible without their efforts.

Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan, Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan

E-mail address: `scchang@math.ntu.edu.tw`