Moduli spaces of real and quaternionc vector bundles over a real algebraic curve

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based on joint work with Florent Schaffhauser (Universidad de Los Andes)

1. Real algebraic curves and Klein surfaces. Let X be an irreducible non-singular projective curve defined over $\mathbb{R}$. Then $M = X(\mathbb{C})$ is a compact connected Riemann surface together with an anti-holomorphic involution $\sigma : M \to M$. The pair (M, σ) is a *Klein surface*. Klein proved that the topological type of a Klein surface (M, σ) is classified by a triple (g, n, a), where $g \in \mathbb{Z}_{\geq 0}$ is the genus of M, $n \in \mathbb{Z}_{\geq 0}$ is the number of connected components of M^σ, the fixed locus of the involution σ, and $a \in \{0, 1\}$ is the index of orientability: $a = 0$ if M/σ is orientable, $a = 1$ if M/σ is nonorientable. M/σ is a compact (orientable or nonorientable) surface (with or without boudary).

2. Real and quaternionic vector bundles. Followng Atiyah, a *real* (resp. *quaternionic*) *holomorphic vector bundle* over a Klein surface (M, σ) is a pair $(\mathcal{E}, \tau)$ with the following properties.

(1) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\tau} & \mathcal{E} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sigma} & M \end{array}$$

(2) $\mathcal{E} \to M$ is a holomorphic vector bundle,
(3) $\tau : \mathcal{E} \to \mathcal{E}$ is anti-holomorphic,
(4) $\tau : \mathcal{E}_x \to \mathcal{E}_{\tau(x)}$ is $\mathbb{C}$-antilinear for any $x \in M$,
(5) $\tau \circ \tau = \mathrm{Id}_E$ (resp. $-\mathrm{Id}_E$).

Similary, one may define a real/quaternionic C^∞ vector bundle (E, τ) over (M, σ).

The topological types of a real/quaternionic C^∞ vector bundle determined by Biswas-Huisman-Hurtubise. Let (M, σ) be a Klein surface of type (g, n, a). When $n > 0$, $M^\sigma = \gamma_1 \cup \cdots \cup \gamma_n$ is a disjoint union of n circles.

($\mathbb{R}$) The topological type of a real vecor bundle $(E, \tau_\mathbb{R}) \to (M, \sigma)$ is classified by $(r, d, w^{(1)}, \ldots, w^{(n)})$, where $r = \mathrm{rank} E \in \mathbb{Z}_{\geq 0}$, $d = \deg E = \int_{[M]} c_1(E) \in \mathbb{Z}$, and $w^{(j)} = w_1(E^{\tau_\mathbb{R}}\big|_{\gamma_j}) \in H^1(\gamma_j, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
Constraint: $w^{(1)} + \cdots w^{(n)} \equiv d \pmod 2$

($\mathbb{H}$) The topological type of a quaternionic vector bundle $(E, \tau_\mathbb{H}) \to (M, \sigma)$ is classified by (r, d)
Constraint: $\begin{cases} d + r(g-1) \equiv 0 \pmod 2, & n = 0 \\ r \equiv d \equiv 0 \pmod 2, & n > 0 \end{cases}$

Let (M, σ) be a Klein surface. Recall that the slope of a holomorphic vector bundle $\mathcal{E}$ over M is $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rank} \mathcal{E}}$. A real/quaternionic holomorphic vector bundle $(\mathcal{E}, \tau)$ over (M, σ) is

(1) *stable* if, for any non-trivial τ-invariant subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$;
(2) *semi-stable* if, for any non-trivial τ-invariant subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$;

(3) *geometrically stable* if, for any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$;
(4) *geometrically semi-stable* if, for any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$.

Apparently, (1)$\Rightarrow$(2), (3)$\Rightarrow$(4), (3)$\Rightarrow$(1), (4)$\Rightarrow$(2). Schaffhauser showed that (2)$\Rightarrow$ (4), (1)$\not\Rightarrow$ (3) (see also Langton).

Following Schaffhauser, let $(\mathcal{E},\tau)$ be a semi-stable real/quaternionic holomorphic vector bundle over (M,σ). A *real/quaternionic Jordan-Hölder filtration* of $(\mathcal{E},\tau)$ is a filtration $0=\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ by τ-invariant holomorphic subbundles, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable in the real/quaternionic sense. Let $\mathrm{gr}(\mathcal{E},\tau) := \bigoplus_{i=1}^k \mathcal{E}_i/\mathcal{E}_{i-1}$. Two semi-stabe real/quaternionic holomorphic vector bundles $(\mathcal{E},\tau)$ and $(\mathcal{E}'.\tau')$ are *real/quaternionic S-equivalent* if $\mathrm{gr}(\mathcal{E},\tau) \cong \mathrm{gr}(\mathcal{E}',\tau')$ as real/quaternionic holomorphic vector bundles.

We fix a C^∞ real/quaternionic vector bundle (E,τ) of rank r, degree d on a Klein surface (M,σ).

- Let $\mathcal{M}^{r,d}_M$ be the moduli space of S-equivalence classes of semi-stable holomorphic structure on E. Atiyah-Bott computed the Poincaré polynomial $P_t(\mathcal{M}^{r,d}_M;\mathbb{Q})$ when $\mathcal{M}^{r,d}_M$ is smooth.
- Let $\mathcal{M}^{r,d,\tau}_{M,\sigma}$ be moduli space of real/quaternionic S-equivalence classes of semi-stable τ-compactible holomoprhic structures on (E,τ). Liu-Schaffhauser computed the Poincaré polynomial $P_t(\mathcal{M}^{r,d,\tau}_{M,\sigma};\mathbb{Z}/2\mathbb{Z})$ when $\mathcal{M}^{r,d,\tau}_{M,\sigma}$ is smooth.

3. The Yang-Mills equations over Riemann surfaces. (Atiyah-Bott, 1982)

Let $E \to M$ be a complex vector bundle of rank r, degree d over a Riemann surface $g \geq 2$. Let $\mathcal{C}$ be the space of $(0,1)$-connections (holomorphic structures) on E. There is a stratificaton $\mathcal{C} = \bigcup_{\mu \in I_{r,d}} \mathcal{C}_\mu$, where $\mathcal{C}_\mu$ is the space of holomorphic structures on E of Harder-Narasimhan type $\mu = (\underbrace{\frac{d_1}{r_1},\ldots,\frac{d_1}{r_1}}_{r_1},\ldots,\underbrace{\frac{d_\ell}{r_\ell},\ldots,\frac{d_\ell}{r_\ell}}_{r_\ell})$,

where $r_i \in \mathbb{Z}_{>0}$, $d_i \in \mathbb{Z}$, $\sum_i r_i = r$, $\sum_i d_i = d$, $\frac{d_1}{r_1} > \cdots > \frac{d_\ell}{r_\ell}$. A holomorphic structure $\mathcal{E}$ is in $\mathcal{C}_\mu$ if $0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$, where $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a semi-stable holomorphic vector bundle of rank r_i, degree d_i over M. Let $\mathcal{C}_{ss} = \mathcal{C}_{(\frac{d}{r},\ldots,\frac{d}{r})}$ be the space of semi-stable holomorphic structure on E.) The gauge group $\mathcal{G}_\mathbb{C} = \mathrm{Aut}(E)$ acts on $\mathcal{C}$, $\mathcal{C}_\mu$, and $\mathbb{C}^* \subset \mathcal{G}_\mathbb{C}$ acts trivially; $\overline{\mathcal{G}}_\mathbb{C} := \mathcal{G}_\mathbb{C}/\mathbb{C}^*$ acts on $\mathcal{C}$, acts freely on $\mathcal{C}_s \subset \mathcal{C}_{ss}$, where $\mathcal{C}_s$ is stable holomorphic structures. When $r \wedge d = 1$, $\mathcal{C}_{ss} = \mathcal{C}_s$, $\mathcal{M}^{r,d}_M = \mathcal{C}_{ss}/\overline{\mathcal{G}}_\mathbb{C}$ is a smooth projective variety over $\mathbb{C}$, and

$$P_t(\mathcal{M}^{r,d}_M;\mathbb{Q}) = P_t^{\overline{\mathcal{G}}_\mathbb{C}}(\mathcal{C}_{ss};\mathbb{Q}) = (1-t^2)P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C}_{ss};\mathbb{Q}).$$

Atiyah-Bott obtained recursive formula for $P_g(r,d) := P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C}_{ss};\mathbb{Q})$ for any $g \geq 2$, $r \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$. Their method can be summarized in the following three steps.

(1) $\{\mathcal{C}_\mu : \mu \in I_{r,d}\}$ is a $\mathcal{G}_\mathbb{C}$-equivariant perfect stratification over $\mathbb{Q}$

$$P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C};\mathbb{Q}) = \sum_{\mu \in I_{r,d}} t^{2d_\mu} P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C}_\mu;\mathbb{Q}), \quad d_\mu = \mathrm{rank}_\mathbb{C} N_{\mathcal{C}_\mu/\mathcal{C}}.$$

Key: $e_{\mathcal{G}_\mathbb{C}}(N_{\mathcal{C}_\mu/\mathcal{C}})$ is not a zero divisor in $H^*_{\mathcal{G}_\mathbb{C}}(\mathcal{C}_\mu;\mathbb{Q})$.

(2) The rational Poincaré series $Q_g(r) := P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C};\mathbb{Q}) = P_t(B\mathcal{G}_\mathbb{C};\mathbb{Q})$ of the classifying space $B\mathcal{G}_\mathbb{C}$ of the gauge group $\mathcal{G}_\mathbb{C}$ can be computed using certain

cohomological Leray-Hirsch spectral sequences over $\mathbb{Q}$. These spectral sequences collapse at the E_2 term.

(3) $\mu = \Big(\underbrace{\frac{d_1}{r_1},\ldots,\frac{d_1}{r_1}}_{r_1},\ldots,\underbrace{\frac{d_\ell}{r_\ell},\ldots,\frac{d_\ell}{r_\ell}}_{r_\ell}\Big) \in I_{r,d} \Rightarrow P_t^{\mathcal{G}_\mathbb{C}}(\mathcal{C}_\mu;\mathbb{Q}) = \prod_{i=1}^\ell P_g(r_i,d_i)$.

(1)+(2)+(3) $\Rightarrow$ Atiyah-Bott recursive formula

$$P_g(r,d) = Q_g(r) - \sum_{\mu\in I_{r,d}-\{(\frac{d}{r},\cdots,\frac{d}{r})\}} t^{2d_\mu}\prod_{i=1}^{\ell} P_t(r_i,d_i)$$

Zagier solved the above recursive formula and obtained a closed formula for $P_g(r,d)$ for any $g\geq 2$, $r\in\mathbb{Z}_{>0}$, $d\in\mathbb{Z}$.

4. The Yang-Mills Equations over Klein Surfaces. (Liu-Schaffhauser, 2011)

Let $(E,\tau)\to(M,\sigma)$ be a real/quaternionic vector bundle of rank r, degree d over a Klein surface (M,σ) of type (g,n,a), where $g\geq 2$. τ induces involutions on $\mathcal{C}$, $\mathcal{C}_\mu$, $\mathcal{G}_\mathbb{C}^\tau$. $\mathcal{G}_\mathbb{C}^\tau$ acts on $\mathcal{C}^\tau$, $\mathcal{C}_\mu^\tau$, and $\mathbb{R}^* = (\mathbb{C}^*)^\tau \subset \mathcal{G}_\mathbb{C}^\tau$ acts trivially. $\overline{\mathcal{G}}_\mathbb{C}^\tau := \mathcal{G}_\mathbb{C}^\tau/\mathbb{R}^*$ acts on $\mathcal{C}^\tau$, acts *freely* on $\mathcal{C}_s^\tau\subset\mathcal{C}_{ss}^\tau$, where $\mathcal{C}_s^\tau$is geometrically stable τ-compatible holomorphic structures When $r\wedge d=1$, $\mathcal{C}_{ss}^\tau=\mathcal{C}_s^\tau$, $\mathcal{M}_{M,\sigma}^{r,d,\tau}=\mathcal{C}_{ss}^\tau/\overline{\mathcal{G}}_\mathbb{C}^\tau$ is a smooth compact manifold, and

$$P_t(\mathcal{M}_{M,\sigma}^{r,d,\tau};\mathbb{Z}_2) = P_t^{\overline{\mathcal{G}}_\mathbb{C}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2) = (1-t)P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2).$$

Liu-Schaffhauser obtained a recursive formula for $P_{(g,n,a)}^{\,\tau}(r,d) := P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2)$ for any (g,n,a), r, d, τ, $(g\geq 2)$. Similar to Atiyah-Bott, our method can be summarized in three steps.

(1) $\{\mathcal{C}_\mu^\tau : \mu\in I_{r,d}^\tau\}$ is a $\mathcal{G}_\mathbb{C}^\tau$-equivariant perfect stratification over $\mathbb{Z}_2$

$$P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}^\tau;\mathbb{Z}_2) = \sum_{\mu\in I_{r,d}^\tau} t^{d_\mu} P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}_\mu^\tau;\mathbb{Z}_2),\quad d_\mu = \mathrm{rank}_\mathbb{R}(N_{\mathcal{C}_\mu^\tau/\mathcal{C}^\tau}).$$

$N_{\mathcal{C}_\mu^\tau/\mathcal{C}^\tau}$ is real vector bundle which is not orientable in general
Key: $e_{\mathcal{G}_\mathbb{C}^\tau}(N_{\mathcal{C}_\mu^\tau/\mathcal{C}^\tau})$ is not a zero divisor in $H^*_{\mathcal{C}_\mathbb{C}}(\mathcal{C}_\mu^\tau;\mathbb{Z}_2)$.

(2) The mod 2 Poincaré series $Q_{(g,n,a)}^{\,\tau}(r) := P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}^\tau;\mathbb{Z}_2) = P_t(B\mathcal{G}_\mathbb{C}^\tau;\mathbb{Z}_2)$ of the classifying space $B\mathcal{G}_\mathbb{C}^\tau$ of the real/quaternionic gauge group $\mathcal{G}_\mathbb{C}^\tau$ can be computed using certain cohomological Leray-Hirsch spectral sequences over $\mathbb{Z}_2$. These spectral sequences do *not* collapse at the E_2 term in general, and we need to compute all the higher differentials.

(3) $\mu = \Big(\underbrace{\frac{d_1}{r_1},\ldots,\frac{d_1}{r_1}}_{r_1},\ldots,\underbrace{\frac{d_\ell}{r_\ell},\ldots,\frac{d_\ell}{r_\ell}}_{r_\ell};\tau_1,\ldots,\tau_\ell\Big) \in I_{r,d}^\tau$

$\Rightarrow P_t^{\mathcal{G}_\mathbb{C}^\tau}(\mathcal{C}_\mu;\mathbb{Z}_2) = \prod_{i=1}^\ell P_{(g,n,a)}^{\,\tau_i}(r_i,d_i)$.

(1)+(2)+(3) $\Rightarrow$ recursive formula

$$P_{(g,n,a)}^{\,\tau}(r,d) = Q_{(g,n,a)}^{\,\tau}(r) - \sum_{\mu\in I_{r,d}^\tau-\{(\frac{d}{r},\cdots,\frac{d}{r})\}} t^{d_\mu}\prod_{i=1}^{\ell} P_{(g,n,a)}^{\,\tau_i}(r_i,d_i).$$

Using Zagier's method, we solved the above recursive formula and obtained a closed formula $P_{(g,n,a)}^{\,\tau}(r,d)$ for any (g,n,a), r, d, τ, $(g\geq 2)$.

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Real algebraic curves and Klein surfaces

Let X be an irreducible nonsingular projective curve defined over $\mathbb{R}$. Then $M = X(\mathbb{C})$ is a compact connected Riemann surface together with an anti-holomorphic involution $\sigma : M \to M$. The pair (M, σ) is a **Klein surface**.

The topological type of a Klein surface

Felix Klein (1893). The topological type of a Klein surface (M, σ) is classified by a triple (g, n, a), where

- $g \in \mathbb{Z}_{\geq 0}$ is the genus of M
- $n \in \mathbb{Z}_{\geq 0}$ is the number of connected components of M^σ
- $a \in \{0, 1\}$ is the index of orientability:

$$a = \begin{cases} 0 & \text{if } M/\sigma \text{ is orientable,} \\ 1 & \text{if } M/\sigma \text{ is nonorientable.} \end{cases}$$

M/σ is a compact (orientable or nonorientable) surface (with or without boudary)

Real and quaternionic vector bundles: definitions

Atiyah (1966). A **real** (resp. **quaternionic**) **holomorphic vector bundle** over a Klein surface (M, σ) is a pair $(\mathcal{E}, \tau)$ with the following properties.

1. There is a commutative diagram
$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\tau} & \mathcal{E} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sigma} & M \end{array}$$
2. $\mathcal{E} \to M$ is a holomorphic vector bundle,
3. $\tau : \mathcal{E} \to \mathcal{E}$ is anti-holomorphic,
4. $\tau : \mathcal{E}_x \to \mathcal{E}_{\tau(x)}$ is $\mathbb{C}$-antilinear for any $x \in M$,
5. $\tau \circ \tau = \mathrm{Id}_E$ (resp. $-\mathrm{Id}_E$).

Similary, one may define a real/quaternionic $\mathbf{C}^\infty$ vector bundle (E, τ) over (M, σ).

Real and quaternionic vector bundles: topological types

Let (M, σ) be a Klein surface of type (g, n, a).
When $n > 0$, $M^\sigma = \gamma_1 \cup \cdots \cup \gamma_n$ disjoint union of n circles
Biswas-Huisman-Hurtubise (2010)

($\mathbb{R}$) The topological type of a real vecor bundle $(E, \tau_{\mathbb{R}}) \to (M, \sigma)$ is classified by $(r, d, w^{(1)}, \ldots, w^{(n)})$, where $r = \mathrm{rank} E \in \mathbb{Z}_{\geq 0}$, $d = \deg E = \int_{[M]} c_1(\mathcal{E}) \in \mathbb{Z}$, and

$$w^{(j)} = w_1(E^{\tau_{\mathbb{R}}}\big|_{\gamma_j}) \in H^1(\gamma_j, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

is the first Stiefel-Whitney class of the real vector bundle $E^{\tau_{\mathbb{R}}}|_{\gamma_j}$ of rank r over the j-th boundary circle γ_j.
Constraint: $w^{(1)} + \cdots w^{(n)} \equiv d \pmod 2$

($\mathbb{H}$) The topological type of a quaternionic vector bundle $(E, \tau_{\mathbb{H}}) \to (M, \sigma)$ is classified by (r, d)
Constraint: $\begin{cases} d + r(g-1) \equiv 0 \pmod 2, & n = 0 \\ r \equiv d \equiv 0 \pmod 2, & n > 0 \end{cases}$

Real and quaternionic vector bundles: stability conditions

Let (M, σ) be a Klein surface. The slope of a holomorphic vector bundle $\mathcal{E}$ over M is $\mu(\mathcal{E}) := \dfrac{\deg \mathcal{E}}{\mathrm{rank}\mathcal{E}}$. A real/quaternionic holomorphic vector bundle $(\mathcal{E}, \tau)$ over (M, σ) is

(1) **stable** if, for any non-trivial τ-invariant subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$;

(2) **semi-stable** if, for any non-trivial τ-invariant subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$;

(3) **geometrically stable** if, for any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$;

(4) **geometrically semi-stable** if, for any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$.

Apparently: (1)$\Rightarrow$(2), (3)$\Rightarrow$(4), (3)$\Rightarrow$(1), (4)$\Rightarrow$(2)
Schaffhauser (2010): (2)$\Rightarrow$ (4), (1)$\not\Rightarrow$ (3) (cf. Langton 1975)

Schaffhauser (2010)

Let $(\mathcal{E},\tau)$ be a semi-stable real/quaternionic holomorphic vector bundle over (M,σ). A **real/quaternionic Jordan-Hölder filbration** of $(\mathcal{E},\tau)$ is a filtration

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_k=\mathcal{E}$$

by τ-invariant holomorphic subbundles, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable in the real/quaternionic sense.

$$\mathrm{gr}(\mathcal{E},\tau):=\bigoplus_{i=1}^{k}\mathcal{E}_i/\mathcal{E}_{i-1}.$$

Two semi-stabe real/quaternionic holomorphic vector bundles $(\mathcal{E},\tau)$ and $(\mathcal{E}'.\tau')$ are **real/quaternionic S-equivalent** if $\mathrm{gr}(\mathcal{E},\tau)\cong\mathrm{gr}(\mathcal{E}',\tau')$ as real/quaternionic holomorphic vector bundles.

Real and quaternionic vector bundles: moduli spaces

We fix a C^∞ real/quaternionic vector bundle (E,τ) of rank r, degree d on a Klein surface (M,σ).

Let $\mathcal{M}_M^{r,d}$ be the moduli space of S-equivalence classes of semi-stable holomorphic structure on E.
Atiyah-Bott (1982) computed the Poincaré polynomial $P_t(\mathcal{M}_M^{r,d};\mathbb{Q})$ when $\mathcal{M}_M^{r,d}$ is smooth.

Let $\mathcal{M}_{M,\sigma}^{r,d,\tau}$ be moduli space of real/quaternionic S-equivalence classes of semi-stable τ-compactible holomoprhic structures on (E,τ).
L-Schaffhauser (2011) computed the Poincaré polynomial $P_t(\mathcal{M}_{M,\sigma}^{r,d,\tau};\mathbb{Z}/2\mathbb{Z})$ when $\mathcal{M}_{M,\sigma}^{r,d,\tau}$ is smooth.

The Yang-Mills equations over Riemann surfaces (I)

Atiyah-Bott (1982) Let $E \to M$ be a complex vector bundle of rank r, degree d over a Riemann surface $g \geq 2$.
$\mathcal{C}$ = space of $(0,1)$-connections (holomorphic structures) on E

$$\mathcal{C} = \bigcup_{\mu \in I_{r,d}} \mathcal{C}_\mu$$

where $\mathcal{C}_\mu$ is the space of holomorphic structures on E of Harder-Narasimhan type $\mu = \Big(\underbrace{\frac{d_1}{r_1}, \ldots, \frac{d_1}{r_1}}_{r_1}, \ldots, \underbrace{\frac{d_\ell}{r_\ell}, \ldots, \frac{d_\ell}{r_\ell}}_{r_\ell}\Big)$,

$r_i \in \mathbb{Z}_{>0}$, $d_i \in \mathbb{Z}$, $\sum_i r_i = r$, $\sum_i d_i = d$, $\dfrac{d_1}{r_1} > \cdots > \dfrac{d_\ell}{r_\ell}$.

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

$\mathcal{E}_i/\mathcal{E}_{i-1}$ semi-stable, rank r_i, degree d_i.

The Yang-Mills equations over Riemann surfaces (II)

$\mathcal{C}_{ss} = \mathcal{C}_{(\frac{d}{r},\dots,\frac{d}{r})}$ = space of semi-stable holomorphic structure on E

The gauge group $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}(E)$ acts on $\mathcal{C}$, $\mathcal{C}_\mu$.
$\mathbb{C}^* \subset \mathcal{G}^{\mathbb{C}}$ acts trivially.
$\overline{\mathcal{G}}_{\mathbb{C}} = \mathcal{G}_{\mathbb{C}}/\mathbb{C}^*$ acts on $\mathcal{C}$, acts freely on $\mathcal{C}_s \subset \mathcal{C}_{ss}$
$\mathcal{C}_s$: stable holomorphic structures

$r \wedge d = 1 \Rightarrow \mathcal{C}_{ss} = \mathcal{C}_s$
$\Rightarrow \mathcal{M}_M^{r,d} = \mathcal{C}_{ss}/\overline{\mathcal{G}}_{\mathbb{C}}$ is a smooth projective variety over $\mathbb{C}$
$P_t(\mathcal{M}_M^{r,d};\mathbb{Q}) = P_t^{\overline{\mathcal{G}}_{\mathbb{C}}}(\mathcal{C}_{ss};\mathbb{Q}) = (1-t^2)P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_{ss};\mathbb{Q})$.

Atiyah-Bott: recursive formula for

$$P_g(r,d) := P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_{ss};\mathbb{Q}) \text{ for any } g \geq 2,\ r \in \mathbb{Z}_{>0},\ d \in \mathbb{Z}$$

The Yang-Mills equations over Riemann surfaces (III)

(1) $\{\mathcal{C}_\mu : \mu \in I_{r,d}\}$ is a $\mathcal{G}_{\mathbb{C}}$-equivariant perfect stratification over $\mathbb{Q}$

$$P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C};\mathbb{Q}) = \sum_{\mu\in I_{r,d}} t^{2d_\mu} P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_\mu;\mathbb{Q}), \quad d_\mu = \mathrm{rank}_{\mathbb{C}} N_{\mathcal{C}_\mu/\mathcal{C}}.$$

Key: $e_{\mathcal{G}_{\mathbb{C}}}(N_{\mathcal{C}_\mu/\mathcal{C}})$ is not a zero divisor in $H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_\mu;\mathbb{Q})$.

(2) $Q_g(r) := P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C};\mathbb{Q}) = P_t(B\mathcal{G}_{\mathbb{C}};\mathbb{Q})$

$$= \frac{\prod_{j=1}(1+t^{2j-1})^{2g}}{\prod_{j=1}^{r}(1-t^{2j})\prod_{j=1}^{r-1}(1-t^{2j})}$$

Method: cohomological Leray-Hirsch spectral sequence over $\mathbb{Q}$. It collapses at the E_2 term.

(3) $\mu = \Big(\underbrace{\frac{d_1}{r_1},\ldots,\frac{d_1}{r_1}}_{r_1},\ldots,\underbrace{\frac{d_\ell}{r_\ell},\ldots,\frac{d_\ell}{r_\ell}}_{r_\ell}\Big) \in I_{r,d}$

$P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_\mu;\mathbb{Q}) = \prod_{i=1}^{\ell} P_g(r_i,d_i)$.

Recursive formula and closed formula

(1)+(2)+(3) $\Rightarrow$ Atiyah-Bott recursive formula

$$P_g(r,d) = Q_g(r) - \sum_{\mu \in I_{r,d} - \{(\frac{d}{r},\cdots,\frac{d}{r})\}} t^{2d_\mu} \prod_{i=1}^{\ell} P_t(r_i,d_i)$$

Zagier's closed formula

$$\begin{aligned} & P_g(r,d) \\ = & \sum_{l=1}^{r} \sum_{\substack{r_1,\dots,r_l \in \mathbb{Z}_{>0} \\ \sum r_i = r}} (-1)^{l-1} \frac{t^{2(\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)\frac{d}{r}\rangle + (g-1)\sum_{i<j} r_i r_j)}}{\prod_{i=1}^{l-1}(1-t^{2(r_i+r_{i+1})})} \\ & \prod_{i=1}^{l} \frac{\prod_{j=1}^{r_i}(1+t^{2j-1})^{2g}}{\left(\prod_{j=1}^{r_i-1}(1-t^{2j})^2\right)(1-t^{2r_i})} \end{aligned}$$

where $\langle x \rangle = [x] + 1 - x$ denotes, for a real number x, the unique $t \in (0,1]$ with $x + t \in \mathbb{Z}$.

The Yang-Mills equations over Klein surfaces (I)

L-Schaffhauser (2011) Let $(E,\tau)\to(M,\sigma)$ be a real/quaternionic vector bundle of rank r, degree d over a Klein surface (M,σ) of type (g,n,a), where $g\geq 2$.

τ induces involutions on $\mathcal{C}$, $\mathcal{C}_\mu$, $\mathcal{G}_{\mathbb{C}}^\tau$.
$\mathcal{G}_{\mathbb{C}}^\tau$ acts on $\mathcal{C}^\tau$, $\mathcal{C}_\mu^\tau$. $\mathbb{R}^*=(\mathbb{C}^*)^\tau\subset\mathcal{G}_{\mathbb{C}}^\tau$ acts trivially.
$\overline{\mathcal{G}}_{\mathbb{C}}^\tau=\mathcal{G}_{\mathbb{C}}^\tau/\mathbb{R}^*$ acts on $\mathcal{C}^\tau$, acts *freely* on $\mathcal{C}_s^\tau\subset\mathcal{C}_{ss}^\tau$
$\mathcal{C}_s^\tau$: geometrically stable τ-compatible holomorphic structures

$r\wedge d=1\Rightarrow \mathcal{C}_{ss}^\tau=\mathcal{C}_s^\tau$
$\Rightarrow \mathcal{M}_{M,\sigma}^{r,d,\tau}=\mathcal{C}_{ss}^\tau/\overline{\mathcal{G}}_{\mathbb{C}}^\tau$ is a smooth compact manifold,

$$P_t(\mathcal{M}_{M,\sigma}^{r,d,\tau};\mathbb{Z}_2)=P_t^{\overline{\mathcal{G}}_{\mathbb{C}}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2)=(1-t)P_t^{\mathcal{G}_{\mathbb{C}}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2).$$

L-Schaffhauser: recursive formula for
$P_{(g,n,a)}^{\tau}(r,d):=P_t^{\mathcal{G}_{\mathbb{C}}^\tau}(\mathcal{C}_{ss}^\tau;\mathbb{Z}_2)$ for any (g,n,a), r, d, τ, $(g\geq 2)$

The Yang-Mills equations over Klein surfaces (II)

(1) $\{\mathcal{C}^\tau_\mu : \mu \in I^\tau_{r,d}\}$ is a $\mathcal{G}^\tau_{\mathbb{C}}$-equivariant perfect stratification over $\mathbb{Z}_2$

$$P_t^{\mathcal{G}^\tau_{\mathbb{C}}}(\mathcal{C}^\tau;\mathbb{Z}_2) = \sum_{\mu\in I^\tau_{r,d}} t^{d_\mu} P_t^{\mathcal{G}^\tau_{\mathbb{C}}}(\mathcal{C}^\tau_\mu;\mathbb{Z}_2), \quad d_\mu = \mathrm{rank}_{\mathbb{R}}(N_{\mathcal{C}^\tau_\mu/\mathcal{C}^\tau}).$$

$N_{\mathcal{C}^\tau_\mu/\mathcal{C}^\tau}$ is real vector bundle which is not orientable in general
Key: $e_{\mathcal{G}^\tau_{\mathbb{C}}}(N_{\mathcal{C}^\tau_\mu/\mathcal{C}^\tau})$ is not a zero divisor in $H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}^\tau_\mu;\mathbb{Z}_2)$.

(2) $Q^\tau_{(g,n,a)}(r) := P_t^{\mathcal{G}^\tau_{\mathbb{C}}}(\mathcal{C}^\tau;\mathbb{Z}_2) = P_t(B\mathcal{G}^\tau_{\mathbb{C}};\mathbb{Z}_2)$
The real case:

$$Q^{\tau_{\mathbb{R}}}_{(g,n,a)}(r) = \frac{\prod_{j=1}^r(1+t^{2j-1})^{g-n+1}\prod_{j=1}^{r-1}(1+t^j)^n\prod_{j=1}^r(1+t^j)^n}{\prod_{j=1}^{r-1}(1-t^{2j})\prod_{j=1}^r(1-t^{2j})}.$$

The Yang-Mills connections over Klein surfaces (III)

The quaternionic case:

n=0

$$Q^{\tau_{\mathbb{H}}}_{(g,0,1)}(r) = \frac{\prod_{j=1}^{r}(1+t^{2j-1})^{g+1}}{\prod_{j=1}^{r-1}(1-t^{2j})\prod_{j=1}^{r}(1-t^{2j})}.$$

$n > 0$ ($\Rightarrow r = 2r'$ even)

$$Q^{\tau_{\mathbb{H}}}_{(g,n,a)}(2r') = \frac{\prod_{j=1}^{2r'}(1+t^{2j-1})^{g}\prod_{j=1}^{r'}(1+t^{4j-1})}{\prod_{j=1}^{2r'-1}(1-t^{2j})\prod_{j=1}^{r'}(1-t^{4j})}.$$

Method: cohomological Leray-Hirsch spectral sequence over $\mathbb{Z}_2$. It does *not* collapose at the E_2 term.

(3) $$\mu = \Big(\underbrace{\frac{d_1}{r_1},\ldots,\frac{d_1}{r_1}}_{r_1},\ldots,\underbrace{\frac{d_\ell}{r_\ell},\ldots,\frac{d_\ell}{r_\ell}}_{r_\ell};\tau_1,\ldots,\tau_\ell\Big) \in I^{\tau}_{r,d}$$

$$P_t^{\mathcal{G}^{\tau}_{\mathbb{C}}}(\mathcal{C}_\mu;\mathbb{Q}) = \prod_{i=1}^{\ell} P^{\tau_i}_{(g,n,a)}(r_i,d_i).$$

Recursive formula and closed formula

(1)+(2)+(3) $\Rightarrow$ recursive formula

$$P^{\tau}_{(g,n,a)}(r,d) = Q^{\tau}_{(g,n,a)}(r) - \sum_{\mu \in I^{\tau}_{r,d} - \{(\frac{d}{r},\cdots,\frac{d}{r})\}} t^{d_\mu} \prod_{i=1}^{\ell} P^{\tau_i}_{(g,n,a)}(r_i,d_i).$$

Closed formula for the $n=0$, real case:

$$P^{\tau_{\mathbb{R}}}_{(g,0,1)}(r,2d)$$

$$= \sum_{l=1}^{r} \sum_{\substack{r_1,\ldots,r_l \in \mathbb{Z}_{>0} \\ \sum r_i = r}} (-1)^{l-1} \frac{t^{2(\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)(\frac{d}{r})\rangle)}}{\prod_{i=1}^{l-1}(1-t^{2(r_i+r_{i+1})})} t^{(g-1)\sum_{i<j} r_i r_j}$$

$$\prod_{i=1}^{l} \frac{\prod_{j=1}^{r_i}(1+t^{2j-1})^{g+1}}{\prod_{j=1}^{r_i}(1-t^{2j}) \prod_{j=1}^{r_i-1}(1-t^{2j})}$$

Closed formula for the $n = 0$, quaternionic case

$$P^{\tau_{\mathbb{H}}}_{(2g'-1,0,1)}(r, 2d)$$

$$= \sum_{l=1}^{r} \sum_{\substack{r_1,\dots,r_l \in \mathbb{Z}_{>0} \\ \sum r_i = r}} (-1)^{l-1} \frac{t^{2\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)(\frac{d}{r})\rangle}}{\prod_{i=1}^{l-1}(1-t^{2(r_i+r_{i+1})})} t^{(2g'-2)\sum_{i<j} r_i r_j}$$

$$\prod_{i=1}^{l} \frac{\prod_{j=1}^{r_i}(1+t^{2j-1})^{2g'}}{\prod_{j=1}^{r_i}(1-t^{2j})\prod_{j=1}^{r_i-1}(1-t^{2j})}$$

$$P^{\tau_{\mathbb{H}}}_{(2g',0,1)}(r, 2d+r)$$

$$= \sum_{l=1}^{r} \sum_{\substack{r_1,\dots,r_l \in \mathbb{Z}_{>0} \\ \sum r_i = r}} (-1)^{l-1} \frac{t^{2\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)(\frac{d}{r})\rangle}}{\prod_{i=1}^{l-1}(1-t^{2(r_i+r_{i+1})})} t^{(2g'-1)\sum_{i<j} r_i r_j}$$

$$\prod_{i=1}^{l} \frac{\prod_{j=1}^{r_i}(1+t^{2j-1})^{2g'+1}}{\prod_{j=1}^{r_i}(1-t^{2j})\prod_{j=1}^{r_i-1}(1-t^{2j})}$$

Closed formula for the $n>0$, real case

$$P^{\tau_{\mathbb{R}}}_{(g,n,a)}(r,d)$$

$$= \sum_{l=1}^{r} \sum_{\substack{r_1,\dots,r_l\in\mathbb{Z}_{>0}\\ \sum r_i=r}} (-1)^{l-1} \frac{t^{\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)(\frac{d}{r})\rangle}}{\prod_{i=1}^{l-1}(1-t^{r_i+r_{i+1}})} t^{(g-1)\sum_{i<j} r_i r_j}$$

$$\cdot 2^{(n-1)(l-1)} \prod_{i=1}^{l} \frac{\prod_{j=1}^{r_i}(1+t^{2j-1})^{g-n+1}\left(\prod_{j=1}^{r_i-1}(1+t^j)^{2n}\right)(1+t^{r_i})^n}{\prod_{j=1}^{r_i}(1-t^{2j})\prod_{j=1}^{r_i-1}(1-t^{2j})}$$

Closed formula for the $n>0$, quaternionic case

$$P^{\tau_{\mathbb{H}}}_{(g,n,a)}(2r,2d)$$

$$= \sum_{l=1}^{r} \sum_{\substack{r_1,\dots,r_l\in\mathbb{Z}_{>0}\\ \sum r_i=r}} (-1)^{l-1} \frac{t^{4\sum_{i=1}^{l-1}(r_i+r_{i+1})\langle (r_1+\cdots+r_i)(\frac{d}{r})\rangle}}{\prod_{i=1}^{l-1}(1-t^{4(r_i+r_{i+1})})} t^{4(g-1)\sum_{i<j} r_i r_j}$$

$$\prod_{i=1}^{l} \frac{\prod_{j=1}^{2r_i}(1+t^{2j-1})^{g}\prod_{j=1}^{r_i}(1+t^{4j-1})}{\prod_{j=1}^{2r_i-1}(1-t^{2j})\prod_{j=1}^{r_i}(1-t^{4j})}$$