New description of the fractional quantum Hall edge

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We have derived a new description of the fractional quantum Hall (FQH) edge: a new chiral Tomonaga-Luttinger liquid theory containing additional interaction terms that represent interaction among chiral bosons in slices of the edge, as an expansion in the noncommutative parameter θ. An important consequence is a correction to the predicted universal exponent of the electron propagator, which has been measured experimentally and found to be nonuniversal. The discrepancy between experiment and old effective Chern-Simon theory is resolved.

First we give a constraint by considering microscopic dynamics: relabeling symmetry of electrons and incompressibility of the fluid.

Consider a 2D electron system, the discrete electrons should be labeled with a discrete index α. Under relabeling (or permutation) of electrons, real space coordinates $x_i^\alpha(t)$ $(i = 1, 2)$ and Lagrangian remain invariant. We can replace α by a continuous y space, and naturally choose the comoving coordinates so that electrons are evenly distributed with a constant density ρ_0. The relabeling symmetry of α is replaced by the area preserving diffeomorphism (APD) of y.

Consider an infinitesimal APD transformation with unit Jacobian, the Lagrangian takes

$$L = \int d^2y\rho_0[\frac{m}{2}\dot{x}^2 - V(\rho) + \frac{eB}{2}\epsilon_{ab}\dot{x}_a x_b], \quad \rho(x,t) = \rho_0|\partial y/\partial x|, \tag{1}$$

$$\delta L = \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_a}\delta x_a) + (-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} + \frac{\partial L}{\partial x_a})\delta x_a = 0. \tag{2}$$

Besides the equation of motion, we arrive at a time independent conserved quantity

$$g(y) = \frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} = |\frac{\partial x}{\partial y}|, \quad \text{if B is very strong.} \tag{3}$$

Consider a FQH fluid with a filling factor $\nu = 1/m$, m is odd. Due to electron-electron (Coulomb) interaction, the electrons are incompressible with a minimal area $2\pi l_B^2/\nu$ ($l_B = \frac{1}{\sqrt{eB}}$ is the magnetic length). (Note that $1/m$ FQH state, placing m vortices onto each electron to reduce Coulomb repulsion, is also APD invariant.) In the absence of vortices (no quasiparticle excitation), we can assume that electrons are in equilibrium and uniformly occupy the minimal area. Thus $\rho(x, 0)$ is space independent, $g(y)$ can be set to unity. The constraint becomes

$$1 = |\frac{\partial x}{\partial y}|. \tag{4}$$

Consider small deviations from equilibrium $x_i = y_i$, dropping total time derivatives and Maxwell term, the Lorentz force gives the Chern-Simons action

$$x_i(y,t) = y_i + \epsilon_{ij}\frac{A_j(y,t)}{2\pi\rho_0} \equiv y_i + \theta\epsilon_{ij}A_j, \quad S = \frac{eB}{2}\int dtd^2y\rho_0\epsilon_{ij}\dot{x}_i x_j = \frac{1}{4\pi\nu}\int dtd^2y\epsilon_{ij}\dot{A}_i A_j. \tag{5}$$

Notice that $\theta = \frac{1}{\nu eB} = l_B^2/\nu$ is a basic area quantum of y space. It means that y space is noncommutative. We involve noncommutativity of y by expanding to higher order in θ.

Second, we solve the constraint exactly. It's amazing that the solution as well as the action has a total differential form.

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The exact solution is $A_i = \sum_{n=0}^{\infty} \theta^n f_i^{(n)}$, $f_i^{(0)} = \partial_i \phi$, $f_i^{(n)} = \frac{1}{2}\epsilon_{ab} \sum_{m=0}^{n-1} \partial_i f_a^{(m)} f_b^{(n-1-m)}$. We construct $F_\mu^{(n)}$ with total differentials and get a conjecture: for all natural numbers, $f_i^{(n)} = F_i^{(n)}$. It has been checked up to $n = 7$, with $F_\mu^{(0)} = \partial_\mu \phi$ ($\mu = 0, 1, 2$, $\partial_0 \equiv \partial_t$) and for $n \geq 1$

$$F_\mu^{(n)} = \partial_a(\frac{1}{2}\epsilon_{ab}F_\mu^{(n-1)}F_b^{(0)}) + \frac{1}{3}\partial_\mu(\frac{1}{2}\epsilon_{ab}F_a^{(n-1)}F_b^{(0)}) - \frac{1}{3}\sum_{m=1}^{n-1}\partial_a(\frac{1}{2}\epsilon_{ab}F_\mu^{(m)}F_b^{(n-1-m)}). \tag{6}$$

Similarly, S is integration of total differentials. It's nonzero and nontrivial iff a boundary exists.

$$S = \frac{1}{4\pi\nu}\int dtd^2y \sum_{n=0}^{\infty}\theta^n s^{(n)}, \quad s^{(n)} = 2F_0^{(n+1)} = \partial_a(\epsilon_{ab}F_0^{(n)}F_b^{(0)}) - \frac{1}{3}\sum_{m=1}^{n}\partial_a(\epsilon_{ab}F_0^{(m)}F_b^{(n-m)}), \tag{7}$$

$$s^{(0)} = \epsilon_{ij}\partial_j[\phi\partial_t\partial_i\phi], \; s^{(1)} = \frac{\epsilon_{ij}\epsilon_{ab}}{3}\partial_j[\partial_t\partial_b\phi\partial_i\phi\partial_a\phi], \; s^{(2)} = \frac{\epsilon_{ij}\epsilon_{ab}\epsilon_{mn}}{4}\partial_j[\partial_t(\partial_i\phi\partial_b\phi)\partial_a\partial_m\phi\partial_n\phi]. \tag{8}$$

Third, we continue with an edge and reduce the $2+1$ dimensional Chern-Simon theory to a $1+1$ dimensional noncommutative interacting chiral Tomonaga-Luttinger liquid theory.

Consider a finite system Σ confined by a simple potential well: an electric field $\vec{E} = E\hat{y}_2$. The electrons drift perpendicular to $\vec{E}$ and B with $v = \frac{E}{B}$ and form an edge. The real space

$$x_i^R = x_i + v_i t = y_i + \theta\epsilon_{ij}A_j + v_i t, \quad S_\Sigma = \int_\Sigma dtd^2y\rho_0(\frac{eB}{2}\epsilon_{ij}\partial_t x_i^R x_j^R - eE_i x_i^R) = S|_\Sigma. \tag{9}$$

In the real laboratory frame $y_i^R = y_i + v_i t$, $t^R = t$, $\partial_t = \partial_t^R + v_i\partial_i^R$, $\partial_i = \partial_i^R$, ignoring R for ease of notation, we can reduce the edge action to a (1+1)D chiral boson theory

$$S_\chi = \frac{1}{4\pi\nu}\int dtdy_1 \sum_{n=0}^{\infty}\theta^n\chi^{(n)}, \quad \chi^{(n)} = -F_0^{(n)}F_1^{(0)} + \frac{1}{3}\sum_{m=1}^{n}F_0^{(m)}F_1^{(n-m)}, \tag{10}$$

the edge velocity $v = \frac{E}{B}$ appears naturally in redefined $F_0^{(0)} = (\partial_t + v\partial_1)\phi$ and for $n \geq 1$

$$F_0^{(n)} = \partial_a(\frac{\epsilon_{ab}}{2}F_0^{(n-1)}F_b^{(0)}) + (\partial_t + v\partial_1)(\frac{\epsilon_{ab}}{6}F_a^{(n-1)}F_b^{(0)}) - \sum_{m=1}^{n-1}\partial_a(\frac{\epsilon_{ab}}{6}F_0^{(m)}F_b^{(n-1-m)}). \tag{11}$$

The commutative limit ($\theta \to 0$) of it coincides with old Chern-Simons and Luttinger theory.

In fact, we get a new chiral Tomonaga-Luttinger liquid theory, which contains interaction among chiral bosons. Interaction makes things different: vertices and loop Feynman diagrams emerge and correct the boson and electron propagators.

Finally, we calculate one-loop Feynman diagrams and get a correction to the exponent.

Following hydrodynamic formulation (bosonization, density-wave and charge excitations),

$$[\frac{1}{2\pi}\partial_1\phi(y_1), \phi(y_1')] = -i\nu\delta(y_1 - y_1'), \quad \Psi \propto e^{i\frac{1}{\nu}\phi}, \quad \Psi(y_1)\Psi(y_1') = (-1)^{1/\nu}\Psi(y_1')\Psi(y_1). \tag{12}$$

With retarded Green's function, 3-boson and 4-boson vertices, $\phi(y) = \int \frac{d^2p}{(2\pi)^2}\tilde{\phi}(p)e^{i(py_1 - \omega_p t)}e^{h(y_2)}$,

$$V_2(p) \equiv \tilde{D}_R(p) = \frac{-i2\pi\nu}{(\omega_p - vp)p}, \quad V_3(p,q) = \frac{-\theta}{4\pi\nu}h'\sum_{l=p,q,r}(\omega_l - vl)l^2, \tag{13}$$

$$V_4(p,q,r) = \frac{i\theta^2}{4\pi\nu}[\frac{1}{4}(h'^2 + 2h'')\sum_{l=p,q,r,k}l^2\sum_{l=p,q,r,k}(\omega_l - vl)l - (h'^2 + h'')\sum_{l=p,q,r,k}(\omega_l - vl)l^3]. \tag{14}$$

Sum over all y_2 slices gives the full boson propagator with one-loop (θ^2) 1PI contributions,

$$\frac{-i2\pi\tilde{\nu}}{p(\omega_p - vp)}. \tag{15}$$

where $c_1 = \frac{\nu}{2\pi}\theta^2\Lambda^2(\frac{3}{2}h'^2 + h'')$, $c_2 = \frac{\nu}{8\pi}\theta^2\Lambda^2 h'^2$, $\tilde{\nu} = \frac{\nu}{1+c_1}$ and $v_n = v(1 - \frac{1+c_1}{c_2}\frac{1}{\ln\Lambda - 1})$. $\Lambda = l_B^{-1}$ is an ultraviolet cutoff, for y space has area quantum θ and shortest incompressible distance l_B. Slightly smaller, v_n is the next-door neighbor of v decreasing along $-y_2$ with damping $\vec{E}$.

We can determine the distribution by solving $c_1 = \frac{l_B^2}{2\pi\nu}(\frac{3}{2}h'^2 + h'')$. As $e^{h(y_2)}$ should decrease along negative y_2 axis, the only solution is $h(y_2) = \sqrt{\frac{4\pi\nu}{3l_B^2}c_1 y_2}$, $c_1 > 0$. Naturally, we choose it and confirm the characteristic length to be radius $\sqrt{2l_B^2/\nu}$. Hence, $c_1 = 3/8\pi$ and $\tilde{\nu} = \frac{\nu}{1+c_1} \approx 0.893\nu$. In the position representation, the boson and electron propagators take the form

$$\langle\phi(y_1,t)\phi(0)\rangle = -\tilde{\nu}\ln(y_1 - vt) + \text{const}, \quad \left\langle T\{\Psi^\dagger(y_1,t)\Psi(0)\}\right\rangle = e^{\frac{\langle\phi(y_1,t)\phi(0)\rangle}{\nu^2}} \propto \frac{1}{(y_1 - vt)^\alpha}, \quad (16)$$

where the tunneling exponent exhibits a new nontrivial power-law

$$\alpha = \frac{\tilde{\nu}}{\nu^2} \approx 0.893\frac{1}{\nu}. \quad (17)$$

At $\nu = 1/3$, the prediction is $\alpha \approx 2.68$. It is in good agreement with the value measured, $\alpha \approx 2.7$. Better than predictions of old effective theory $\alpha = 3$.

Strictly, the formalism only applies to Laughlin fractions. If we abandon Fermi statistics, we can extend the calculation to Jain fractions $\nu = 2/5$ and $3/7$ giving $\alpha \approx 2.23$ and 2.08, close to the measured exponents 2.3 and 2.1. It is interesting but further precise study is needed.

In summary, we develop a new description to describe the FQH edge better.

Furthermore, the edge of integer quantum Hall effect to be described with Maxwell term, of hierarchial liquids and other related topics are remained as future subjects.