

Weak Mirror Symmetry of Complex Symplectic Algebras

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Abstract

A complex symplectic structure on a Lie algebra \mathfrak{h} is an integrable complex structure J with a closed non-degenerate $(2, 0)$ -form. It is determined by J and the real part Ω of the $(2, 0)$ -form. Suppose that \mathfrak{h} is a semi-direct product $\mathfrak{g} \ltimes V$, and both \mathfrak{g} and V are Lagrangian with respect to Ω and totally real with respect to J . This note shows that $\mathfrak{g} \ltimes V$ is its own weak mirror image in the sense that the associated differential Gerstenhaber algebras controlling the extended deformations of Ω and J are isomorphic.

The geometry of (Ω, J) on the semi-direct product $\mathfrak{g} \ltimes V$ is also shown to be equivalent to that of a torsion-free flat symplectic connection on the Lie algebra \mathfrak{g} . By further exploring a relation between (J, Ω) with hypersymplectic algebras, we find an inductive process to build families of complex symplectic algebras of dimension $8n$ from the data of the $4n$ -dimensional

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1. Introduction

In this paper, we continue our analysis on weak mirror symmetry of Lie algebras with a focus on complex symplectic structures.

Recall that the extended deformation theory of complex or symplectic structures are dictated by their associated differential Gerstenhaber algebras (a.k.a. DGA) and the induced Gerstenhaber algebra structure on its cohomology ring [4] [15]. If M is a manifold with a complex structure J and M^\vee is another manifold with a symplectic structure ω with the same dimension, then (M, J) and (M^\vee, ω) form a weak mirror pair if the associated differential Gerstenhaber algebras $\text{DGA}(M, J)$ and $\text{DGA}(M^\vee, \omega)$ are quasi-isomorphic [12].

Inspired by the concept of T-duality in mirror symmetry [14] [5], we develop a set of tools to analyze weak mirror symmetry on semi-direct product Lie algebras [7], and apply it to study complex and symplectic structures on real six-dimensional nilpotent algebras in details [8] [7].

However, the work in [7] does not explain an elementary, but non-trivial example on Kodaira-Thurston surface, a real four-dimensional example [13]. Yet Kodaira-Thurston surface is a key example of hyper-symplectic structures [11] [9]. As explained by Hitchin, from a G -structure perspective, hyper-symplectic structures are the non-compact counterpart of hyper-Kähler structures. The unifying feature is a complex symplectic structure, i.e. a $\bar{\partial}$ -closed holomorphic non-degenerate $(2, 0)$ -form on complex even-dimensional manifolds [10].

In this notes, we analyze complex symplectic structures adopted to a semi-direct product Lie algebra.

Our first observation is Proposition 4.3, which essentially states that if \mathfrak{g} is a Lie algebra symplectic structure ω and a torsion-free, flat, symplectic connection γ , then the semi-direct product $\mathfrak{g} \ltimes_\gamma V$ admits a complex symplectic structure Ω such that \mathfrak{g} and V are totally real, and Lagrangian with respect to the real part of Ω . Moreover, every complex symplectic structure

on a Lie algebra with such characteristics arises in exactly this way. Algebras with such complex symplectic structures are called *special Lagrangian*.

With a computation using dual representation and the results in [8] and [7], we derive the main result of this notes in Theorem 5.2, which states that a semi-direct product Lie algebra, special Lagrangian with respect to the complex symplectic structure is its own weak mirror image.

It is known that hypersymplectic structures form a special sub-class of complex symplectic structure. In Section 6, we explore this relation, and find in Theorem 6.1 that whenever one gets a special Lagrangian complex symplectic Lie algebra, one gets a tower of such objects in higher dimension.

In the last section, we produce a few examples explicitly. In particular, we put the work of the second author on Kodaira-Thurston within the framework of complex symplectic structures and provide a theoretical framework to explain the ad hoc computation in [13]. A set of new examples generalizing the hypersymplectic structure on Kodaira-Thurston surfaces is constructed.

2. Summary of Weak Mirror Symmetry

We briefly review the basic definitions and results in [8] and [7]. Except for key concepts, details are referred to these two papers.

It is known that for a manifold M with a complex structure J there associates a differential Gerstenhaber algebra $DGA(M, J)$ over \mathbb{C} . This object dictates the deformation theory of the complex structure. For a manifold N with a symplectic form Ω , there also associates a differential Gerstenhaber algebra structure $DGA(N, \Omega)$. It dictates its deformation theory in an extended sense [4]. If M and N are manifolds of the same dimension, and if $DGA(M, J)$ and $DGA(N, \omega)$ are quasi-isomorphic, then the pair of structures (M, J) and (N, Ω) are said to be a weak mirror pair [7] [12]. We focus on the case when the manifolds in questions are Lie groups, and all geometric structures are left-invariant. Therefore, all constructions are expressed in terms of Lie algebras. In particular, exterior differential becomes the Chevalley-Eilenberg differentials on the dual of a Lie algebra. In this notes, if $DGA(1)$ and $DGA(2)$ are two quasi-isomorphic differential Gerstenhaber algebras, we denote such a relation by $DGA(1) \sim DGA(2)$. The notation $DGA(1) \cong DGA(2)$ denotes isomorphism.

Suppose \mathfrak{h} is the semi-direct product of a subalgebra \mathfrak{g} and an abelian ideal V : $\mathfrak{h} = \mathfrak{g} \ltimes V$. The semi-direct product structure is said to be totally

real with respect to an integrable complex structure J on \mathfrak{h} if

$$J\mathfrak{g} = V \quad \text{and} \quad JV = \mathfrak{g}.$$

It is said to be Lagrangian with respect to a symplectic form Ω on \mathfrak{h} if both \mathfrak{g} and V are Lagrangian. By contraction in the first variable, a two-form on \mathfrak{h} is a linear map from \mathfrak{h} to \mathfrak{h}^* . It is non-degenerate if the induced map is a linear isomorphism. If the semi-direct product is Lagrangian, then

$$\Omega : \mathfrak{g} \rightarrow V^* \quad \text{and} \quad \Omega : V \rightarrow \mathfrak{g}^*.$$

Given a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes V$, the adjoint action of \mathfrak{g} on V is a representation of \mathfrak{g} on V . Conversely, given any presentation $\gamma : \mathfrak{g} \rightarrow \text{End}(V)$, one could define a Lie bracket on the vector space $\mathfrak{g} \oplus V$ by

$$[(x, u), (y, v)] = ([x, y], \gamma(x)v - \gamma(y)u), \quad (1)$$

where $x, y \in \mathfrak{g}$ and $u, v \in V$. The resulting Lie algebra is denoted by $\mathfrak{g} \ltimes_{\gamma} V$. One could take the dual representation $\gamma^* : \mathfrak{g} \rightarrow \text{End}(V^*)$: for any v in V , α in V^* and x in \mathfrak{g} ,

$$(\gamma^*(x)\alpha)v := -\alpha(\gamma(x)v).$$

We address the Lie algebra $\widehat{\mathfrak{h}} := \mathfrak{g} \ltimes_{\gamma^*} V^*$ as the dual semi-direct product of $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$.

In [7], it is noted that weak mirror pairs on semi-direct products are given in terms of dual semi-direct products.

Proposition 2.1. [7] *Let $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ be a semi-direct product totally real with respect to an integrable complex structure J . On the dual semi-direct product $\widehat{\mathfrak{h}} = \mathfrak{g} \ltimes_{\gamma^*} V^*$, define*

$$\widehat{\Omega}((x, \mu), (y, \nu)) := \nu(Jx) - \mu(Jy), \quad (2)$$

where $x, y \in \mathfrak{g}$, $\mu, \nu \in V^*$. Then $\widehat{\Omega}$ is a symplectic form, and the dual semi-direct product $\widehat{\mathfrak{h}}$ is Lagrangian with respect to $\widehat{\Omega}$. Moreover, there is a natural isomorphism $\text{DGA}(\mathfrak{h}, J) \cong \text{DGA}(\widehat{\mathfrak{h}}, \widehat{\Omega})$.

Proposition 2.2. [7] *Let $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ be a semi-direct product Lagrangian with respect to a symplectic form Ω . On the dual semi-direct product $\widehat{\mathfrak{h}} = \mathfrak{g} \ltimes_{\gamma^*} V^*$, define*

$$\widehat{J}(x, \mu) := (-\Omega^{-1}(\mu), \Omega(x)), \quad (3)$$

where $x \in \mathfrak{g}$ and $\mu \in V^*$. Then \widehat{J} is an integrable complex structure, and the dual semi-direct product $\widehat{\mathfrak{h}}$ is totally real with respect to \widehat{J} . Moreover, there is a natural isomorphism $\text{DGA}(\mathfrak{h}, \Omega) \cong \text{DGA}(\widehat{\mathfrak{h}}, \widehat{J})$.

3. Complex Symplectic Algebras

In this section, we present a few elementary facts on complex symplectic structures on Lie algebras. One may globalize the geometry to Lie groups by left translation. In favorable situation, the geometry descends to a co-compact quotient. Therefore, lots of consideration in this section has its counterpart on manifolds. A discussion on the relation between complex symplectic structures, hyperKähler structures and hypersymplectic structures could be found in [10]. We stay focus on Lie algebras.

Definition 3.1. [6], [10] *A complex symplectic structure on a real Lie algebra \mathfrak{h} is a pair (J, Ω_c) where J is an integrable complex structure and Ω_c is a non-degenerate closed $(2, 0)$ -form.*

Let Ω_1 and Ω_2 be the real and imaginary parts of Ω_c respectively, $\Omega_c = \Omega_1 + i\Omega_2$. It is immediate that they are closed real two-forms on \mathfrak{h} . Define

$$\mathfrak{h}^{1,0} = \{X - iJX \in \mathfrak{h}_{\mathbb{C}} : X \in \mathfrak{h}\}, \quad \mathfrak{h}^{0,1} = \{X + iJX \in \mathfrak{h}_{\mathbb{C}} : X \in \mathfrak{h}\}.$$

They are respectively the $+i$ and $-i$ eigenspace of the complex structure on the complexification of \mathfrak{h} .

Since the complex structure J is integrable, $\bar{\partial}\Omega_c = 0$ and $\partial\Omega_c = 0$. By definition of non-degeneracy, the map

$$\Omega_c : \mathfrak{h}^{1,0} \rightarrow \mathfrak{h}^{*(1,0)}$$

is a linear isomorphism.

Lemma 3.2. *On a real vector space with an almost complex structure J the real part Ω_1 and imaginary part Ω_2 of a non-degenerate complex $(2, 0)$ -form Ω_c are non-degenerate two-forms with the properties*

$$\Omega_2(X, Y) = -\Omega_1(JX, Y); \tag{4}$$

$$\Omega_1(JX, Y) = \Omega_1(X, JY), \quad \Omega_2(JX, Y) = \Omega_2(X, JY). \tag{5}$$

Conversely, given any real non-degenerate two-form Ω satisfying (5), let $\Omega_1 = \Omega$ and define Ω_2 by (4), then $\Omega_c = \Omega_1 + i\Omega_2$ is a non-degenerate $(2, 0)$ -form.

Proof: Suppose Ω_c is a $(2,0)$ -form on a real vector space W with almost complex structure J for any real vectors X and Y , $\Omega_c(X + iJX, Y) = 0$. Setting $\Omega_c = \Omega_1 + i\Omega_2$, and taking the real and imaginary part, one get the relation in (4).

Since Ω_1 is real, $\Omega_1(X + iJX, Y + iJY) = \Omega_1(X - iJX, Y - iJY)$. The identity (5) for Ω_1 follows. An identical computation shows that Ω_2 satisfies (5).

Conversely, given a real two-form Ω_1 satisfying (5). Define Ω_2 by (4), then Ω_2 satisfies (5). It follows that $\Omega_c := \Omega_1 + i\Omega_2$ is a $(2,0)$ -form.

Consider the non-degeneracy of Ω_c . $\Omega_c(X - iJX) = 0$ if and only if

$$\Omega_1(X) + \Omega_2(JX) = 0 \quad \text{and} \quad \Omega_1(JX) - \Omega_2(X) = 0.$$

By (4), $\Omega_1(X) = \Omega_2(JX)$. Therefore, $\Omega_c(X - iJX) = 0$ if and only if $\Omega_1(X) = 0$ and $\Omega_2(X) = 0$. Therefore, Ω_c is non-degenerate if and only if both Ω_1 and Ω_2 are non-degenerate. \blacksquare

On a real Lie algebra \mathfrak{h} with complex structure J , a $(2,0)$ -form $\Omega_c = \Omega_1 + i\Omega_2$ is closed if and only if Ω_1 and Ω_2 are closed.

Given the complex structure, the Chevalley-Eilenberg differential d has a type decomposition. Denote its $(1,0)$ -part by ∂ . Its $(0,1)$ -part by $\bar{\partial}$. The integrability of J implies that $d\Omega_c = 0$ if and only if $\partial\Omega_c = 0$ and $\bar{\partial}\Omega_c = 0$.

Since $d\Omega_1 = 0$, for any X, Y, Z in \mathfrak{h} ,

$$\bar{\partial}\Omega_1(X + iJX, Y - iJY, Z - iJZ) = 0.$$

As a consequence of the vanishing Nijenhuis tensor and (5), the real and imaginary part of the above constraints respectively

$$\begin{aligned} \Omega_1([X, Y], Z) + \Omega_1(J[JX, Y], Z) \\ + \Omega_1([Z, X], Y) + \Omega_1(J[Z, JX], Y) = 0. \end{aligned} \quad (6)$$

$$\begin{aligned} \Omega_1([X, Y], JZ) + \Omega_1(J[JX, Y], JZ) \\ + \Omega_1([JZ, X], Y) + \Omega_1(J[JZ, JX], Y) = 0. \end{aligned} \quad (7)$$

The latter is apparently equivalent to the former as the map J is a real linear isomorphism from \mathfrak{h} to itself.

Lemma 3.3. *Let \mathfrak{h} be a Lie algebra with an integrable complex structure J . Let Ω_1 be a symplectic form on \mathfrak{h} satisfying (5). Define $\Omega_2 = -\Omega_1 \circ J$, then Ω_2 is a symplectic form and satisfies (5).*

Proof: Let X, Y, Z be any vectors in \mathfrak{h} . By integrability of J ,

$$\begin{aligned}
d\Omega_2(X, Y, Z) &= \\
&= -\Omega_1(J[X, Y], Z) - \Omega_1(J[Y, Z], X) - \Omega_1(J[Z, X], Y) \\
&= \Omega_1([JX, Y], Z) + \Omega_1([X, JY], Z) - \Omega_1([JX, JY], JZ) \\
&\quad + \Omega_1([JY, Z], X) + \Omega_1([Y, JZ], X) - \Omega_1([JY, JZ], JX) \\
&\quad + \Omega_1([JZ, X], Y) + \Omega_1([Z, JX], Y) - \Omega_1([JZ, JX], JY).
\end{aligned}$$

Since $d\Omega_1 = 0$, the above is equal to

$$\begin{aligned}
&\Omega_1([JX, Y], Z) + \Omega_1([Z, JX], Y) - \Omega_1([Z, X], JY) - \Omega_1([X, Y], JZ) \\
&= -\Omega_1(J[JX, Y], JZ) - \Omega_1([X, Y], JZ) + \Omega_1([Z, JX], Y) - \Omega_1([Z, X], JY).
\end{aligned}$$

By (7), it is equal to

$$\Omega_1([JZ, X] + J[JZ, JX] + [Z, JX] - J[Z, X], Y),$$

which is equal to zero by integrability of J . ■

Proposition 3.4. *Given a complex structure J on a Lie algebra \mathfrak{h} , a two-form $\Omega_c = \Omega_1 + i\Omega_2$ is non-degenerate and closed if and only if Ω_1 is a symplectic form satisfying (5) and $\Omega_2 = -\Omega_1 \circ J$.*

Conversely, if Ω_1 is a symplectic form on \mathfrak{h} satisfying (5), define $\Omega_2 := -\Omega_1 \circ J$, then $\Omega_1 + i\Omega_2$ is a complex symplectic structure.

Proof: The first part of this proposition is a paraphrase of Lemma 3.2.

Conversely, given $\Omega_1 = \Omega$. Define Ω_2 by (4). Since the complex structure is integrable, and Ω_1 satisfies (5), $d\Omega_1 = 0$ implies that Ω_1 satisfies (6). As a consequent of Lemma 3.3 $\Omega_2 = -\Omega_1 \circ J$ is a symplectic form. ■

4. Special Lagrangian Structures

As a consequence of Proposition 3.4, whenever J is a complex structure and Ω is a *real* symplectic form on a Lie algebra \mathfrak{h} such that (5) holds, the pair (J, Ω) is addressed as a complex symplectic structure. However, one should note that if Ω_c is a complex symplectic form with respect to J , so is $re^{-i\theta}\Omega_c$ for any $r > 0$ and $\theta \in [0, 2\pi)$. While the real factor r will only

change the symplectic forms Ω_1 and Ω_2 by real homothety, the factor $e^{-i\theta}$ changes Ω_1 to $\cos \theta \Omega_1 + \sin \theta \Omega_2$. Therefore, the geometry related to a choice of Ω_1 or Ω is not invariant of the complex homothety on Ω_c . As we choose to work with a particular choice of Ω_1 within complex homothety, straightly speaking, we work with *polarized* complex symplectic structures.

We examine the characterization of this type of objects when the underlying algebra is a semi-direct product. A reason for working with semi-direct product is due to the fact that the underlying Gerstenhaber algebra $\text{DGA}(\mathfrak{h}, J)$ of a Lie algebra \mathfrak{h} with a complex structure J is the exterior algebra of a semi-direct product Lie algebra. Therefore, when one searches for a weak mirror image for (\mathfrak{h}, J) , it is natural to search among semi-direct products. Details for this discussion could be found in [7]. In our current investigation, the geometry and algebra together also force upon us semi-direct product structures.

Lemma 4.1. *Let \mathfrak{h} denote a Lie algebra equipped with a complex symplectic structure (J, Ω) . Assume $V \subset \mathfrak{h}$ is an isotropic ideal. Then V is abelian; and JV is a subalgebra. Moreover if V is totally real and Lagrangian then $\mathfrak{h} = JV \ltimes V$.*

Proof: Let $x, y \in V$. The closeness condition of Ω says for all z in V ,

$$0 = \Omega([x, y], z) + \Omega([y, z], x) + \Omega([z, x], y) = \Omega([x, y], z).$$

Since Ω is non-degenerate, $[x, y] = 0$ for all $x, y \in V$. It proves that V is abelian.

Using the fact that V is abelian and the integrability condition of J one gets

$$[Jx, Jy] = J([Jx, y] + [x, Jy]) \quad \forall x, y \in V$$

which proves that JV is a subalgebra.

If V is Lagrangian, then JV is also Lagrangian. Moreover whenever V is totally real then $V \cap JV = \{0\}$ and so one gets $\mathfrak{h} = JV \ltimes V$. \blacksquare

Definition 4.2. *Let \mathfrak{h} be a real Lie algebra, V an abelian ideal in \mathfrak{h} and \mathfrak{g} a complementary subalgebra so that $\mathfrak{h} = \mathfrak{g} \ltimes V$. We say that a complex symplectic structure (J, Ω) on \mathfrak{h} is special Lagrangian with respect to the semi-direct product if \mathfrak{g} and V are totally real with respect to J and isotropic with respect to Ω .*

The rest of this section is to characterize the geometry of a special Lagrangian structure (J, Ω) on $\mathfrak{g} \times V$ in terms of a connection on \mathfrak{g} .

The semi-direct product is given by the restriction of the adjoint representation of \mathfrak{h} to \mathfrak{g} . It becomes a representation ρ of the Lie subalgebra \mathfrak{g} on the vector space V . There follows a torsion-free flat connection γ associated to the complex structure J [7]. For any x, y in \mathfrak{g} ,

$$\gamma(x)y := -J\rho(x)Jy. \quad (8)$$

Let $\Omega_1 = \Omega$ be a two-form on \mathfrak{h} in Proposition 3.4. It satisfies (7). Apply this formula to x, y, Jz when $x, y, z \in \mathfrak{g}$. Since Jz, Jx are in V , and V is abelian, $[Jz, Jx] = 0$. By definition of γ and the vanishing of its torsion, Identity (7) is reduced to

$$\begin{aligned} 0 &= \Omega(\gamma(x)y - \gamma(y)x, Jz) + \Omega(\gamma(y)x, Jz) + \Omega(\gamma(x)z, Jy) \\ &= \Omega(\gamma(x)y, Jz) - \Omega(\gamma(x)z, Jy). \end{aligned}$$

As $\Omega_2 = -\Omega \circ J$. The above is equivalent to

$$\Omega_2(\gamma(x)y, z) - \Omega_2(\gamma(x)z, y) = \Omega_2(\gamma(x)y, z) + \Omega_2(y, \gamma(x)z) = 0. \quad (9)$$

Note that Ω_2 is a symplectic form on \mathfrak{h} . If $\Omega_2(x, y) = \Omega_1(Jx, y) = 0$ for all $x \in \mathfrak{g}$, then $y = 0$ because $J\mathfrak{g} = V$ and Ω_1 is isotropic on \mathfrak{g} . Therefore, the restriction of Ω_2 on \mathfrak{g} is non-degenerate. So is the restriction of Ω_2 on V . It follows that Ω_2 is restricted to symplectic forms on \mathfrak{g} and V . Due to Identity (5) and the assumption on J being totally real, the symplectic structure on V is completely dictated by the restriction of Ω_2 on \mathfrak{g} . Denote this restriction on \mathfrak{g} by ω . Then (9) is equivalent to the symplectic form ω on \mathfrak{g} being parallel with respect to the connection γ . i.e.

$$\omega(\gamma(x)y, z) + \omega(y, \gamma(x)z) = 0. \quad (10)$$

In summary, γ is a torsion-free flat symplectic connection on \mathfrak{g} with respect to the symplectic form ω .

Conversely, suppose \mathfrak{g} is equipped with a symplectic form ω , parallel with respect to a flat torsion-free connection γ on \mathfrak{g} . Let V be the underlying vector space of \mathfrak{g} . Define $\mathfrak{h} := \mathfrak{g} \times_{\gamma} V$, set

$$J(x, y) = (-y, x), \quad (11)$$

then J is an integrable complex structure. \mathfrak{g} and V are totally real. Note that in this case, for $x, y \in \mathfrak{g}$,

$$\begin{aligned} -J\gamma(x, 0)J(y, 0) &= -J\gamma(x, 0)(0, y) = -J(0, \gamma(x)y) = -(-\gamma(x)y, 0) \\ &= (\gamma(x)y, 0). \end{aligned}$$

If we use the notation x to denote $(x, 0)$ in \mathfrak{h} , then the above identity becomes

$$-J\gamma(x)Jy = \gamma(x)y.$$

Therefore, one may consider the representation γ playing both the role of representation ρ and the role of the connection γ in (8). This coincidence is the consequence of V being the underlying vector space of the Lie algebra \mathfrak{g} .

Next, define

$$\Omega_1((x, u), (y, v)) := -\omega(x, v) - \omega(u, y). \quad (12)$$

$$\Omega_2((x, u), (y, v)) := \omega(x, y) - \omega(u, v). \quad (13)$$

It is apparent that Ω_1 and Ω_2 are non-degenerate skew-symmetric form on \mathfrak{h} , with \mathfrak{g} and V isotropic with respect to Ω_1 . Moreover,

$$\begin{aligned} \Omega_1(J(x, y), J(x', y')) &= \Omega_1((-y, x), (-y', x')) = -\omega(-y, x') - \omega(x, -y') \\ &= -\Omega_1((x, y), (x', y')). \\ \Omega_2(J(x, y), J(x', y')) &= \Omega_2((-y, x), (-y', x')) = \omega(-y, -y') - \omega(x, x') \\ &= -\Omega_2((x, y), (x', y')). \\ -\Omega_1((x, y), J(x', y')) &= -\Omega_1((x, y), (-y', x')) = \omega(x, x') + \omega(y, -y') \\ &= \Omega_2((x, y), (x', y')). \end{aligned}$$

Therefore, (4) and (5) are satisfied.

$$\begin{aligned} &-d\Omega_1((x_1, y_1), (x_2, y_2), (x_3, y_3)) \\ &= \Omega_1([(x_1, y_1), (x_2, y_2)], (x_3, y_3)) + \text{cyclic permutation} \\ &= \Omega_1([x_1, x_2], \gamma(x_1)y_2 - \gamma(x_2)y_1, (x_3, y_3)) + \text{cyclic permutation} \\ &= \omega([x_1, x_2], y_3) + \omega(\gamma(x_1)y_2 - \gamma(x_2)y_1, x_3) + \text{cyclic permutation} \\ &= \omega(\gamma(x_1)x_2, y_3) - \omega(\gamma(x_2)x_1, y_3) + \omega(\gamma(x_1)y_2, x_3) - \omega(\gamma(x_2)y_1, x_3) \\ &\quad + \text{cyclic permutation}. \end{aligned}$$

It is equal to zero because ω is parallel with respect to γ (10). Therefore, Ω_1 is symplectic. Ω_2 is also a symplectic form due to $d\omega = 0$ and (10). Therefore, $\Omega_c = \Omega_1 + i\Omega_2$ is closed, and it defines a complex symplectic structure. Below is a summary of our observations in this section.

Proposition 4.3. *Let $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ be a semi-direct product algebra. Suppose that it admits $(\Omega_1 + i\Omega_2, J)$ as a special Lagrangian complex symplectic structure. Let ω be the restriction of Ω_2 on \mathfrak{g} . Then ω is a symplectic structure on the Lie algebra \mathfrak{g} , γ is a torsion-free, flat connection on \mathfrak{g} preserving the symplectic structure ω .*

Conversely, given a Lie algebra \mathfrak{g} with a symplectic structure ω and a torsion-free flat symplectic connection, the semi-direct product $\mathfrak{g} \ltimes_{\gamma} V$ admits a complex symplectic structure (J, Ω) such that both \mathfrak{g} and V are totally real with respect to J and Lagrangian with respect to Ω .

5. The Dual Semi-direct Product

Let $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ be a semi-direct product with a special Lagrangian complex holomorphic structure (J, Ω) . Recall that $\gamma : \mathfrak{g} \rightarrow \text{End}(V)$ is a representation. It induces a dual representation: $\gamma^* : \mathfrak{g} \rightarrow \text{End}(V^*)$.

Use x, y, z and a, b, c to represent generic elements in \mathfrak{g} . α, β to represent generic elements in \mathfrak{g}^* , u, v, w to represent generic elements in V , μ, ν, ζ to represent generic elements in V^* . By Proposition 4.3, the restriction of

$$\omega(x, y) := -\Omega(Jx, y) \tag{14}$$

is a symplectic form on \mathfrak{g} . By contractions, ω defines an isomorphism.

Lemma 5.1. *Let V be the underlying symplectic vector space of Lie algebra \mathfrak{g} . The linear map $\varpi : \mathfrak{g} \ltimes_{\gamma} V \rightarrow \mathfrak{g} \ltimes_{\gamma^*} V^*$ defined by*

$$\varpi(x, u) := (x, \omega(u)) \tag{15}$$

is a Lie algebra isomorphism.

Proof: By definition of the semi-direct product $\mathfrak{g} \ltimes_{\gamma^*} V^*$,

$$[(x, \omega(u)), (y, \omega(v))] = ([x, y], \gamma^*(x)\omega(v) - \gamma^*(y)\omega(u)).$$

For all $c \in V$,

$$\begin{aligned} & \left(\gamma^*(x)\omega(v) - \gamma^*(y)\omega(u) \right)(c) \\ &= -\omega(v)(\gamma(x)c) + \omega(u)(\gamma(y)c) = -\omega(v, \gamma(x)c) + \omega(u, \gamma(y)c). \end{aligned}$$

Since γ is a symplectic connection, the last quantity is equal to

$$\begin{aligned} & \omega(\gamma(x)v, c) - \omega(\gamma(y)u, c) = \omega(\gamma(x)v - \gamma(y)u, c) \\ & = \omega(\gamma(x)v - \gamma(y)u)c. \end{aligned}$$

$$\text{As } [(x, u), (y, v)] = ([x, y], \gamma(x)v - \gamma(y)u),$$

$$[(x, \omega(u)), (y, \omega(v))] = ([x, y], \omega(\gamma(x)v - \gamma(y)u)).$$

■

On $\mathfrak{g} \ltimes_{\gamma^*} V^*$, define \tilde{J} and $\tilde{\Omega}$ by

$$\tilde{J} := \varpi \circ J \circ \varpi^{-1}, \quad \varpi^* \tilde{\Omega} = \Omega. \quad (16)$$

Since ϖ is an isomorphism of Lie algebra, \tilde{J} and $\tilde{\Omega}$ defines a complex symplectic structure on $\widehat{\mathfrak{h}} = \mathfrak{g} \ltimes_{\gamma^*} V^*$. Explicitly,

$$\tilde{J}(x, \omega(u)) = \varpi(J(x, u)) = \varpi(Ju, Jx) = (Ju, \omega(Jx)). \quad (17)$$

$$\tilde{\Omega}((x, \omega(u)), (y, \omega(v))) = \Omega((x, u), (y, v)) = \Omega(x, v) + \Omega(u, y). \quad (18)$$

Also, define $\tilde{\Omega}_2 := -\tilde{\Omega} \circ \tilde{J}$. By (17) and (18),

$$\begin{aligned} & \tilde{\Omega}_2((x, \omega(u)), (y, \omega(v))) = -\tilde{\Omega}_1(\tilde{J}(x, \omega(u)), (y, \omega(v))) \\ & = -\tilde{\Omega}((Ju, \omega(Jx)), (y, \omega(v))) = -\Omega(Ju, v) - \Omega(Jx, y) \\ & = -\Omega(J(x, u), (y, v)) = \Omega_2((x, u), (y, v)). \end{aligned}$$

The restriction of $\tilde{\Omega}_2$ on the summand \mathfrak{g} in $\mathfrak{g} \ltimes_{\gamma^*} V^*$ is identical to the restriction of the Ω_2 on $\mathfrak{g} \ltimes_{\gamma} V$. It is the symplectic two-form ω .

By Proposition 2.1, given a totally real complex structure J on $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$, there exists a symplectic form $\widehat{\Omega}$ on $\widehat{\mathfrak{h}}$. It is determined by (2). By (14), $\omega(x) = -\Omega(Jx)$. i.e. $\omega(Jx) = \Omega(x)$.

$$\begin{aligned} & \widehat{\Omega}((x, \omega(u)), (y, \omega(v))) = \omega(v)(Jx) - \omega(u)(Jy) \\ & = \omega(v, Jx) - \omega(u, Jy) = -\Omega(Jv, Jx) + \Omega(Ju, Jy) \\ & = \Omega(v, x) - \Omega(u, y). \end{aligned}$$

Comparing with (18), we find that $\widehat{\Omega} = -\widetilde{\Omega}$. Now we obtain isomorphisms of differential Gerstenhaber algebras:

$$\begin{aligned} \mathrm{DGA}(\mathfrak{h}, J) &\cong \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\Omega}) && \text{by Proposition 2.1} \\ &= \mathrm{DGA}(\widehat{\mathfrak{h}}, \widetilde{\Omega}) && \text{due to } \widehat{\Omega} = -\widetilde{\Omega}, \\ &\cong \mathrm{DGA}(\mathfrak{h}, \Omega) && \text{by (16) and Lemma 5.1.} \end{aligned}$$

Similarly, by Proposition 2.2, if the semi-direct product \mathfrak{h} is Lagrangian with respect to the symplectic form Ω , there exists a complex structure \widehat{J} on $\widehat{\mathfrak{h}}$. It is determined by (3).

$$\begin{aligned} \widehat{J}(x, \omega(u)) &= (-\Omega^{-1}(\omega(u)), \Omega(x)) = (\Omega^{-1}(\Omega(Ju)), \omega(Jx)) \\ &= (Ju, \omega(Jx)). \end{aligned}$$

Comparing with (17), we find that $\widehat{J} = \widetilde{J}$. Therefore, we again obtain the isomorphisms:

$$\begin{aligned} \mathrm{DGA}(\mathfrak{h}, \Omega) &\cong \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J}) && \text{by Proposition 2.2} \\ &= \mathrm{DGA}(\widehat{\mathfrak{h}}, \widetilde{J}) && \text{due to } \widehat{J} = \widetilde{J} \\ &\cong \mathrm{DGA}(\mathfrak{h}, J) && \text{by (16) and Lemma 5.1.} \end{aligned}$$

The last two strings of isomorphisms yield the main observation in this notes.

Theorem 5.2. *Every special Lagrangian complex symplectic semi-direct product is isomorphic to its mirror image.*

To be precise, let $\mathfrak{h} = \mathfrak{g} \ltimes V$ be a semi-direct product, J an integrable complex structure, $\Omega_c = \Omega_1 + i\Omega_2$ a closed $(2, 0)$ -form such that \mathfrak{h} is totally real with respect to J and Lagrangian with respect to $\Omega = \Omega_1$, then $\mathfrak{h} \cong \widehat{\mathfrak{h}}$ as Lie algebras, and $\mathrm{DGA}(\mathfrak{h}, J) \cong \mathrm{DGA}(\mathfrak{h}, \Omega) \cong \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J}) \cong \mathrm{DGA}(\widehat{\mathfrak{h}}, \widetilde{\Omega})$.

6. Relation to Hypersymplectic Geometry

Given a special Lagrangian complex symplectic Lie algebra $\mathfrak{h} = \mathfrak{g} \ltimes_{\rho} V$ with (Ω, J) , there are more structures naturally tied to the given ones. For instance, consider the linear map $E : \mathfrak{g} \ltimes_{\rho} V \rightarrow \mathfrak{g} \ltimes_{\rho} V$ defined by $E \equiv \mathbf{1}$ on \mathfrak{g} and $E \equiv -\mathbf{1}$ on V . Then E is an example of an *almost product structure* [3]. Since both \mathfrak{g} and V are Lie subalgebras of \mathfrak{h} , this almost product structure is integrable, or a product structure without qualification. Since $JV = \mathfrak{g}$

and $J\mathfrak{g} = V$, $JE = -EJ$. Therefore, the pair $\{J, E\}$ forms an example of a complex product structure [3].

Given any complex product structure, there exists a unique torsion-free connection Γ on \mathfrak{h} with respect to which J and E are parallel. For x, y, x', y' in \mathfrak{g} , define

$$\Gamma_{(x+Jx')}(y + Jy') := -J\rho(x)Jy + \rho(x)Jy'. \quad (19)$$

In terms of the direct sum decomposition $\mathfrak{h} = \mathfrak{g} \times_{\rho} V$, we use $(x, 0)$ to denote an element x in \mathfrak{g} . Then for any $y \in \mathfrak{g}$, $Jy = (0, y)$. It follows that the connection with respect to the direct sum decomposition is given by

$$\Gamma_{(x,x')}(y, y') := (\rho(x)y, \rho(x)y'). \quad (20)$$

We check that this is a torsion-free connection.

$$\begin{aligned} & \Gamma_{(x+Jx')}(y + Jy') - \Gamma_{(y+Jy')}(x + Jx') - [x + Jx', y + Jy'] \\ = & (-J\rho(x)Jy + \rho(x)Jy') - (-J\rho(y)Jx + \rho(y)Jx') - ([x, y] + \rho(x)Jy' - \rho(y)Jx') \\ = & -J\rho(x)Jy + J\rho(y)Jx - [x, y] \\ = & \gamma(x)y - \gamma(y)x - [x, y]. \end{aligned}$$

Therefore, Γ is torsion-free if and only if the connection γ is torsion-free.

To compute ΓJ , we find

$$\begin{aligned} & \Gamma_{(x+Jx')}J(y + Jy') - J\Gamma_{(x+Jx')}(y + Jy') \\ = & \Gamma_{(x+Jx')}(-y' + Jy) - J(-J\rho(x)Jy + \rho(x)Jy') \\ = & J\rho(x)Jy' + \rho(x)Jy - \rho(x)Jy - J\rho(x)Jy' = 0. \end{aligned}$$

Therefore, $\Gamma J = 0$. Similarly, $\Gamma E = 0$ because

$$\begin{aligned} & \Gamma_{(x+Jx')}E(y + Jy') - E\Gamma_{(x+Jx')}(y + Jy') \\ = & \Gamma_{(x+Jx')}(y - Jy') - E(-J\rho(x)Jy + \rho(x)Jy') \\ = & -J\rho(x)Jy - \rho(x)Jy' + J\rho(x)Jy - \rho(x)Jy' = 0. \end{aligned}$$

Note that $\Gamma E = 0$ implies that the connection Γ preserves the eigenspaces \mathfrak{g} and V . Therefore, one may consider the restriction of Γ onto the respective subspaces. Indeed, the restriction of Γ onto the $+1$ eigenspace \mathfrak{g} is precisely the connection γ as seen in (8).

Recall that the real and imaginary parts of the complex symplectic form are two different real symplectic forms. In a direct sum decomposition, they

are respectively given by (12) and (13). We now consider a third differential form.

$$\Omega_3((x, u), (y, v)) = \omega(x, y) + \omega(u, v). \quad (21)$$

The three differential forms are related by the endomorphisms J and E :

$$\Omega_1((x, u), (y, v)) = \Omega_2(J(x, u), (y, v)), \quad (22)$$

$$\Omega_3((x, u), (y, v)) = \Omega_2(E(x, u), (y, v)). \quad (23)$$

Treating the contraction with the first entry in any 2-form as an endomorphism from a vector space to its dual, we have

$$\Omega_1 = \Omega_2 \circ J, \quad \Omega_3 = \Omega_2 \circ E, \quad \Omega_1 = \Omega_3 \circ J \circ E, \quad (24)$$

$$\Omega_2 = -\Omega_1 \circ J, \quad \Omega_2 = \Omega_3 \circ E, \quad \Omega_3 = -\Omega_1 \circ E \circ J. \quad (25)$$

Define a bilinear form g on \mathfrak{h} by

$$g((x, u), (y, v)) := \Omega_2(JE(x, u), (y, v)) = -\omega(x, v) + \omega(u, y). \quad (26)$$

It is a non-degenerate symmetric bilinear form on a $4n$ -dimensional vector space with signature $(2n, 2n)$, also known as a neutral metric. Since

$$J \circ J = -\mathbf{I}, \quad E \circ E = \mathbf{I}, \quad (J \circ E) \circ (J \circ E) = \mathbf{I}, \quad J \circ E = -J \circ E, \quad (27)$$

where \mathbf{I} is the identity map, we could express the three 2-forms in terms of the neutral metric g and the complex product structures $\{J, E, JE\}$.

$$\Omega_1(\bullet, \bullet) = g(E\bullet, \bullet), \quad \Omega_2(\bullet, \bullet) = g(JE\bullet, \bullet), \quad \Omega_3(\bullet, \bullet) = g(J\bullet, \bullet). \quad (28)$$

By [2, Proposition 5], $d\Omega_2 = 0$ implies that $d\Omega_3 = 0$. Also $d\Omega_2 = 0$ and only if $d\Omega_1 = 0$. Given the complex symplectic structure, Ω_1 and Ω_2 are closed. Therefore, Ω_3 is also closed. It follows that $(g, \Omega_1, \Omega_2, \Omega_3)$ forms a hypersymplectic structure. In addition, by [2, Corollary 10], the connection Γ is the Levi-Civita connection for the neutral metric g . Since $\Gamma J = 0$ and $\Gamma E = 0$, it follows that

$$\Gamma\Omega_1 = \Gamma\Omega_2 = \Gamma\Omega_3 = 0. \quad (29)$$

In addition, since $\Gamma_{Jx} = 0$ for all $x \in \mathfrak{g}$, and it is only a matter of tracing definitions to verify that

$$\Gamma_x \Gamma_{x'} - \Gamma_{x'} \Gamma_x - \Gamma_{[x, x']} = 0.$$

The connection Γ is flat. As a consequence of the second part of Proposition 4.3, we have the following.

Theorem 6.1. *Let \mathfrak{g} be a symplectic Lie algebra with a torsion-free, flat symplectic connection γ on the underlying vector space V of the Lie algebra. Then the space $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ admits a hypersymplectic structure $\{\Omega_1, \Omega_2, \Omega_3\}$ such that the Levi-Civita connection Γ of the associated neutral metric is flat, and symplectic with respect to each of the three given symplectic structures.*

In particular, each of these three symplectic structures induces a complex symplectic structure on $\mathfrak{h} \ltimes_{\Gamma} W$ where W is the underlying vector space of the algebra \mathfrak{h} .

7. Examples

In this section, we present explicit computations. Theorem 6.1 of the previous section provide further examples for exploration.

To recap, (\mathfrak{g}, ω) denote a real Lie algebra equipped with a symplectic structure. Let V denote the underlying vector space of \mathfrak{g} . We seek a linear map $\gamma : \mathfrak{g} \rightarrow \text{End}(V)$ such that for all $x, y, z \in \mathfrak{g}$, it is

- torsion-free: $\gamma(x)y - \gamma(y)x = [x, y]$;
- symplectic: $\omega(\gamma(x)y, z) + \omega(y, \gamma(x)z) = 0$;
- flat: $\gamma([x, y]) = \gamma(x)\gamma(y) - \gamma(y)\gamma(x)$.

The last two conditions are equivalent to require the map γ to admit a symplectic representation of \mathfrak{g} on (V, ω) :

$$\gamma : \mathfrak{g} \rightarrow \mathfrak{sp}(V, \omega). \quad (30)$$

On \mathfrak{g} , choose a basis $\{e_1, \dots, e_{2n}\}$. Then the dual basis is denoted by $\{e^1, \dots, e^{2n}\}$ such that the symplectic form is

$$\omega = e^1 \wedge e^2 + \dots + e^{2n-1} \wedge e^{2n}.$$

Let V be the underlying symplectic vector space for the Lie algebra \mathfrak{g} . The corresponding basis and dual basis are respectively $\{v_1, \dots, v_{2n}\}$ and $\{v^1, \dots, v^{2n}\}$. Let γ_{jk}^l and c_{jk}^l be the structural constants for the representation γ and the algebra \mathfrak{g} :

$$\gamma(e_j)e_k = \gamma_{jk}^l e_l, \quad \text{and} \quad [e_j, e_k] = c_{jk}^l e_l.$$

Then the structure equations for the semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_{\gamma} V$ are

$$[e_j, e_k] = c_{jk}^l e_l, \quad [e_j, v_k] = \gamma_{jk}^l v_l. \quad (31)$$

The complex structure J is determined by $Je_j = v_j$ as dictated by (11). The three real symplectic forms on the semi-direct product \mathfrak{h} are given by (12), (13) and (21) respectively:

$$\Omega_1 = \sum_{j=1}^n (-e^{2j-1} \wedge v^{2j} + e^{2j} \wedge v^{2j-1}); \quad (32)$$

$$\Omega_2 = \sum_{j=1}^n (e^{2j-1} \wedge e^{2j} - v^{2j-1} \wedge v^{2j}). \quad (33)$$

$$\Omega_3 = \sum_{j=1}^n (e^{2j-1} \wedge e^{2j} + v^{2j-1} \wedge v^{2j}). \quad (34)$$

7.1. Four-dimensional examples

There are only two two-dimensional Lie algebras, the abelian one \mathbb{R}^2 and the algebra $\mathfrak{aff}(\mathbb{R})$ of affine transformations on \mathbb{R} .

Consider \mathbb{R}^2 as an abelian Lie algebra with its canonical symplectic structure $\omega = e^1 \wedge e^2$.

Proposition 7.1. [1] *All torsion-free, flat, symplectic connection on the abelian algebra \mathbb{R}^2 is equivalent to $(\mathbb{R}^2, \gamma, e^1 \wedge e^2)$, where γ is either trivial ($\gamma = 0$), or*

$$\gamma(e_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \gamma(e_2) = 0.$$

As a consequence, the resulting four-dimensional complex symplectic Lie algebras are either abelian or isomorphic to the trivial one-dimensional extension of the three-dimensional Heisenberg Lie algebra. We make further change of notations to facilitate future computations. With respect to a vector space decomposition of $\mathfrak{g} = \mathbb{R}^2 \ltimes_{\gamma} \mathbb{R}^2$, a basis is given by

$$f_1 = (e_1, 0) = e_1, \quad f_2 = (e_2, 0) = e_2, \quad f_3 = (0, e_1) = v_1, \quad f_4 = (0, e_2) = v_2.$$

Then the non-trivial example given in Proposition 7.1 above implies that $\gamma(e_1)f_3 = f_4$ on the semi-direct product. The structure equation for \mathfrak{g} is given by

$$[f_1, f_3] = (0, \gamma(e_1)e_1) = (0, e_2) = f_4. \quad (35)$$

The two symplectic forms on \mathfrak{g} are respectively

$$\Omega_1 = -f^1 \wedge f^4 + f^2 \wedge f^3, \quad \Omega_2 = f^1 \wedge f^2 - f^3 \wedge f^4. \quad (36)$$

Moreover, the complex structure J on \mathfrak{g} is

$$Jf_1 = f_3, \quad Jf_2 = f_4. \quad (37)$$

Here we recover the complex and symplectic structures studied in [13], and hence the related claim on the self-mirror property [13, Theorem 19].

In the context of hypersymplectic structure, the third 2-form is

$$\Omega_3 = f^1 \wedge f^2 + f^3 \wedge f^4. \quad (38)$$

Consider next $\mathfrak{aff}(\mathbb{R})$ the Lie algebra spanned by e_1, e_2 with the Lie bracket $[e_1, e_2] = e_2$, with symplectic structure $\omega = e^1 \wedge e^2$.

Proposition 7.2. [1] *All torsion-free, flat, symplectic connection on the algebra of group of affine transformations $\mathfrak{aff}(\mathbb{R})$ is equivalent to $(\mathfrak{aff}(\mathbb{R}), \gamma, e^1 \wedge e^2)$, where γ is one of the following.*

- $\gamma(e_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma(e_2) = 0$.
- $\gamma(e_1) = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\gamma(e_2) = \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix}$.

With these connections on $\mathfrak{aff}(\mathbb{R})$, one applies Proposition 4.3 to construct two complex symplectic algebras in real-dimension four.

Corollary 7.3. *Suppose that $\mathfrak{g} \ltimes_{\gamma} V$ is a non-abelian four-dimensional semi-direct product such that it is special Lagrangian with respect to (J, Ω) , a complex symplectic structure, then it is one of three cases. With respect to a basis $\{f_1, f_2, f_3, f_4\}$, the structure equations are respectively*

- $[f_1, f_3] = f_4$,
- $[f_1, f_2] = f_2$, $[f_1, f_3] = -f_3$, $[f_1, f_4] = f_4$.
- $[f_1, f_2] = f_2$, $[f_1, f_3] = -\frac{1}{2}f_3$, $[f_1, f_4] = \frac{1}{2}f_4$, $[f_2, f_3] = -\frac{1}{2}f_4$.

The complex structure is determined by $Je_1 = e_3$, $Je_2 = e_4$. The complex symplectic form is $\Omega_c = \Omega_1 + i\Omega_2 = i(f^1 + if^3) \wedge (f^2 + if^4)$. where $\Omega = \Omega_1 = -f^1 \wedge f^4 + f^2 \wedge f^3$ and $\Omega_2 = f^1 \wedge f^2 - f^3 \wedge f^4$.

7.2. An example of Theorem 6.1

To construct an explicit example for Theorem 6.1, we consider the canonically induced structure on the non-trivial semi-direct product extending $\mathfrak{h} := \mathbb{R}^2 \ltimes_{\gamma} \mathbb{R}^2$ as given in Proposition 7.1. On this space, we choose a basis in terms of the direct sum $\mathfrak{h} \oplus W$ where W is the underlying vector space of \mathfrak{h} :

$$f_j = (f_j, 0), \quad v_j = (0, f_j), \quad \text{for } 1 \leq j \leq 4. \quad (39)$$

By (20), a torsion-free, flat, symmetric connection Γ on the direct sum $\mathfrak{h} \oplus W$ is given by

$$\Gamma_{f_j} = \begin{pmatrix} \gamma(f_j) & 0 \\ 0 & \gamma(f_j) \end{pmatrix}, \quad \Gamma_{v_j} = 0. \quad (40)$$

Since the only non-trivial term is given by $\gamma(f_1)f_3 = f_4$, the non-trivial terms for Γ are given by

$$\Gamma_{f_1}f_3 = f_4, \quad \Gamma_{f_1}v_3 = v_4. \quad (41)$$

They also determine the structural equations for the algebra $\mathfrak{h} \ltimes_{\Gamma} W$:

$$[f_1, f_3] = f_4, \quad [f_1, v_3] = v_4. \quad (42)$$

This algebra is eight-dimensional. Its center is five-dimensional, spanned by f_2, f_4, v_1, v_2, v_4 . Since the commutator $C(\mathfrak{h}) = \text{span}\{f_4, v_4\}$ is contained in the center, \mathfrak{h} is a 2-step nilpotent Lie algebra.

Work in Section 6 shows that the connection Γ is torsion-free, flat and symplectic with respect to the symplectic forms Ω_1, Ω_2 in (36) and Ω_3 in (38). By Theorem 6.1, any real linear combination of these symplectic form determines a complex symplectic structure on the real eight-dimensional algebra $\mathfrak{h} \ltimes_{\Gamma} W$.

7.3. 2-step nilmanifolds with hypersymplectic structures

We shall produce 2-step nilpotent Lie algebras \mathfrak{h} carrying hypersymplectic structures, generalizing the first Lie algebra in Corollary 7.3.

Let ω denote the canonical symplectic structure on \mathbb{R}^{2n} :

$$\omega = \sum_{i=1}^n e^i \wedge e^{i+n}. \quad (43)$$

Let A_1, \dots, A_n be $n \times n$ -symmetric matrices. Let $\gamma : \mathbb{R}^{2n} \rightarrow \mathfrak{sp}(2n, \omega)$ be the linear map represented, in the basis e_1, e_2, \dots, e_{2n} , by the matrices:

$$\gamma(e_i) = \begin{pmatrix} 0 & 0 \\ A_i & 0 \end{pmatrix}, \quad \gamma(e_{i+n}) = 0, \quad \text{for all } i = 1, 2, \dots, n.$$

Treating \mathbb{R}^{2n} as the trivial abelian algebra, then γ is a representation. Notice that $[\gamma(e_i), \gamma(e_j)] = 0$ so that γ gives rise to a flat, symplectic connection on \mathbb{R}^{2n} .

The torsion-free condition on \mathbb{R}^{2n} requires $\gamma(e_i)e_j = \gamma(e_j)e_i$ for all $i, j = 1, 2, \dots, n$. It is equivalent to require that the j -th column of A_i is equal to the i -th column of A_j . Hence for $n = 1$ one gets

$$\gamma(e_1) = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad \gamma(e_2) = 0.$$

It compares with the non-trivial connection in Proposition 7.1. For $n = 2$

$$\gamma(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ b & c & 0 & 0 \end{pmatrix}, \quad \gamma(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & c & 0 & 0 \\ c & d & 0 & 0 \end{pmatrix}, \quad \gamma(e_3) = 0 = \gamma(e_4),$$

where all coefficients in the matrices above are real numbers.

The resulting Lie algebra $\mathfrak{h} = \mathbb{R}^{2n} \ltimes_{\gamma} \mathbb{R}^{2n}$ is a 2-step nilpotent Lie algebra. In fact, take $e_1, e_2, \dots, e_{2n}, v_1, v_2, \dots, v_{2n}$ as a basis of \mathfrak{h} . Then the non-trivial Lie bracket relations are given by

$$[e_i, v_j] = \sum_{k=1}^n a_{ij}^k v_{n+k} \tag{44}$$

for all $1 \leq i, j \leq n$, where for each i (a_{ij}^k) is the matrix A_i . As the commutator of \mathfrak{h} is generated by $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$, which belongs to the center of \mathfrak{h} , \mathfrak{h} is 2-step nilpotent. It is non-abelian if at least one of the matrices A_i is not trivial. By Theorem 6.1, \mathfrak{h} carries a hypersymplectic structure.

If all the coefficients a_{ij}^k are rational numbers, the simply connected Lie group H with Lie algebra \mathfrak{h} admits a co-compact lattice. The resulting space may be considered as a generalization of the Kodaira-Thurston surface, as seen from the perspective of a hypersymplectic manifold in [11].

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