Fluid Partial Differential Equations with Partial or Fractional Dissipation

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This talk presents some recent results on two systems of PDEs with partial or fractional dissipation:

1) the 2D incompressible Boussinesq equations; and

2) the 2D incompressible magnetohydrodynamic equations.
2D Boussinesq

The standard 2D Boussinesq equations can be written as

\[
\begin{align*}
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} + \theta \mathbf{k}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\theta_t + \mathbf{u} \cdot \nabla \theta &= \kappa \Delta \theta
\end{align*}
\]

where \( \mathbf{u} \) denotes the 2D velocity field, \( p \) the pressure, \( \theta \) the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, \( \nu \) the viscosity, \( \kappa \) the thermal diffusivity, and \( \mathbf{k} \) is the unit vector in the vertical direction.
The 2D Boussinesq equation can be regarded as a special case of the 3D rotating Boussinesq equations. The 3D rotating Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see the books by A.E. Gill, J. Pedlosky, A. Majda and others). In addition, the Boussinesq equations also play an important in the study of Rayleigh-Benard convection (see, e.g., P. Constantin and C. Doering, etc).
Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows. The fundamental issue of whether classical solutions to the 3D Euler and Navier-Stokes equations can develop finite time singularities remains outstandingly open and the study of the 2D Boussinesq equations may shed light on this extremely challenging problem.
In order to model anisotropic Boussinesq flows (the viscosity and the thermal diffusivity are different in the horizontal and vertical directions), (1) should be written as

\[
\begin{align*}
    u_t + uu_x + vu_y &= -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\
    v_t + uv_x + vv_y &= -p_y + \nu_1 v_{xx} + \nu_2 v_{yy} + \theta, \\
    u_x + v_y &= 0, \\
    \theta_t + u\theta_x + v\theta_y &= \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy},
\end{align*}
\]

(2)

where \( u \) and \( v \) are the horizontal and vertical components of \( \overrightarrow{u} \), \( \nu_1 \geq 0, \nu_2 \geq 0, \kappa_1 \geq 0 \) and \( \kappa_2 \geq 0 \). (2) is called anisotropic Boussinesq equations.
A different type of generalization is to replace the Laplacian by fractional Laplacians, namely

$$
\begin{align*}
\vec{u}_t + \vec{u} \cdot \nabla \vec{u} &= -\nabla p - \nu (\Delta)^\alpha \vec{u} + \theta \vec{k}, \\
\nabla \cdot \vec{u} &= 0, \\
\theta_t + \vec{u} \cdot \nabla \theta &= -\kappa (\Delta)^\beta \theta
\end{align*}
$$

(3)

where $0 < \alpha, \beta \leq 1$, and the fractional Laplacian can be defined by the Fourier transform (or through Riesz potential),

$$
(\hat{\Delta})^\alpha \hat{f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).
$$

This system will be called fractional Boussinesq equations.
We consider the initial-value problems (IVPs) of the equations in (2) and in (3) with the initial data

\[ \overrightarrow{u}(x, y, 0) = \overrightarrow{u}_0(x, y), \quad \theta(x, y, 0) = \theta_0(x, y). \]

Do these IVPs have a global solution for sufficiently smooth data \((\overrightarrow{u}_0, \theta_0)\)?

Attention will be focused on either whole space or periodic domain case. There are results for bounded domains (see the papers of M. Lai, R. Pan and K. Zhao, ARMA, 2011 and of K. Zhao).
The global regularity problem on the 2D Boussinesq equations has attracted considerable attention recently from the PDE community. Here is a partial list of people who have worked on it: H. Abidi, D. Adhikari, J.R. Cannon, C. Cao, D. Chae, P. Constantin, R. Danchin, E. DiBenedetto, C. Doering, D. Li, W. E, T. Hmidi, T. Hou, D. KC, S. Keraani, S.-K. Kim, A. Larios, M. Lai, C. Li, E. Lunasin, C. Miao, H.-S. Nam, M. Paicu, R. Pan, D. Regmi, F. Rousset, C. Shu, L. Tao, E.S. Titi, V. Vicol, J. Wu, X. Xu, L. Xue, Z. Zhang, K. Zhao, ···.
One general idea for proving the global (in time) existence and uniqueness. This is divided into two steps:

1) Local existence and uniqueness. This is in general done by the contraction mapping principle for

\[ f(t) = G(f(t)) \equiv f(0) + \int_0^t N(f(\tau)) \, d\tau. \]

This usually requires that the time interval is small.

2) Global bounds and the extension theorem. The nonlinear term is treated as bad part and the dissipation as good part.
Consider the following seven cases:

- $\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0$, inviscid Boussinesq
- $\nu_1 > 0$, $\nu_2 > 0$, $\kappa_1 > 0$ and $\kappa_2 > 0$, dissipation and thermal diffusion
- $\nu_1 > 0$, $\nu_2 > 0$, $\kappa_1 = \kappa_2 = 0$. dissipation but no thermal diffusion
- $\kappa_1 > 0$, $\kappa_2 > 0$, $\nu_1 = \nu_2 = 0$. thermal diffusion but no dissipation
- $\nu_1 > 0$
  horizontal dissipation only

- $\kappa_1 > 0$
  horizontal thermal diffusion only

- $\nu_2 > 0$ and $\kappa_2 > 0$
  vertical dissipation and vertical thermal diffusion
Inviscid Boussinesq, $\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0$

Inviscid Boussinesq equation:

\[
\begin{aligned}
\vec{u}_t + \vec{u} \cdot \nabla \vec{u} &= -\nabla p + \theta \vec{k}, \\
\nabla \cdot \vec{u} &= 0, \\
\theta_t + \vec{u} \cdot \nabla \theta &= 0
\end{aligned}
\]

The vorticity $\omega = \nabla \times \vec{u}$ satisfies

\[
\omega_t + \vec{u} \cdot \nabla \omega = \partial_{x_1} \theta.
\]

The global regularity problem remains open. In fact, the inviscid Boussinesq can be identified with the 3D axisymmetric Euler with swirl.
Recall the 3D axisymmetric Euler equations

\[ u_\theta^t + \left( u^r \partial_r + u^z \partial_z \right) u^\theta + \frac{u^r u^\theta}{r} = 0, \quad \tilde{D} \left( r u^\theta \right) = 0, \quad \tilde{D} \left( r u^\theta \right)^2 = 0 \]

where \( \tilde{D} \) = \( \partial_t + (u^r \partial_r + u^z \partial_z) \). The swirl component of the vorticity \( \omega^\theta \) satisfies

\[ \tilde{D} \left( \frac{\omega^\theta}{r} \right) = -\frac{1}{r^4} \partial_z (r u^\theta)^2 \]

While the 2D Boussinesq is given by, with \( \frac{D}{Dt} = \partial_t + \vec{u} \cdot \nabla \)

\[ \frac{D}{Dt} \theta = 0, \quad \frac{D}{Dt} \omega = \partial_{x_1} \theta \]
We have local existence and regularity criteria.

**Theorem**

Given \((\overrightarrow{u}_0, \theta_0) \in H^s(\mathbb{R}^2)\) with \(s > 2\). Then there exists a unique local classical solution \((\overrightarrow{u}, \theta) \in C([0, T_0); H^s)\) for some \(T_0 > 0\). In addition, if

\[
\int_0^T \| \nabla u(\cdot, t) \|_\infty \, dt < \infty \quad \text{or} \quad \int_0^T \| \nabla \theta(\cdot, t) \|_\infty \, dt < \infty
\]

for \(T > T_0\), then the solution remains in \(H^s\) for any \(t \leq T\).
Does
\[ \int_0^T \| \omega(\cdot, t) \|_\infty \, dt < \infty \]
imply the global regularity? This does not appear to be trivial. A work by C. Cao, P. Constantin, A. Kiselev and J. Wu is in progress.

For the 3D Euler equation, \( u_t + u \cdot \nabla u = -\nabla p, \nabla \cdot u = 0 \), we have the Beale-Kato-Majda criterion, which says that if
\[ \int_0^T \| \omega(\cdot, t) \|_\infty \, dt < \infty, \]
then the solution remains regular on \([0, T]\).
**Remark.** It is possible to prove small data global well-posedness for some weakly damped inviscid Boussinesq.


This type of results are not completely trivial since the $L^2$-norm of the velocity may grow in time.
Viscous Boussinesq equations: the global regularity is well-known

\[
\begin{align*}
    u_t + uu_x + \nu u_y &= -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\
    \nu_t + \nu u_x + \nu v_y &= -p_y + \nu_1 \nu_{xx} + \nu_2 \nu_{yy} + \theta, \\
    u_x + v_y &= 0, \\
    \theta_t + u\theta_x + \nu \theta_y &= \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy},
\end{align*}
\]

(J. E. Cannon & E. DiBenedetto 80; Charles Li 04)

**Theorem**

Let \((\vec{u}_0, \theta_0) \in H^1(\mathbb{R}^2)\). Then there exists a unique global strong solution \((\vec{u}, \theta)\) satisfying, for any \(T > 0\),

\[
\vec{u}, \theta \in L^\infty([0, T; H^1(\mathbb{R}^2)]) \cap L^2([0, T]; H^2(\mathbb{R}^2))
\]
The 2D Boussinesq equations with no thermal diffusion:

\[
\begin{align*}
    u_t + uu_x + \nu u_y &= -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\
    \nu_t + \nu u_x + \nu \nu_y &= -p_y + \nu_1 \nu_{xx} + \nu_2 \nu_{yy} + \theta, \\
    u_x + \nu_y &= 0, \\
    \theta_t + u \theta_x + \nu \theta_y &= 0,
\end{align*}
\]

H.K. Moffatt listed this global regularity issue as one of the 21st century problems. The global regularity holds for this case.
T. Hou & Congming Li 05, DCDSA; D. Chae 06, Adv. Math.

**Theorem**

Let $\nu_1 > 0$ and $\nu_2 > 0$ be fixed. Let $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$. Then there exists a unique solution $(\mathbf{u}, \theta)$ satisfying

$$
\mathbf{u} \in C([0, \infty); H^s) \cap L^2(0, T; H^{s+1}), \quad \theta \in C([0, \infty); H^s).
$$
Thermal Diffusion only, $\nu_1 = \nu_2 = 0$, $\kappa_1 > 0$, $\kappa_2 > 0$

The 2D Boussinesq equations with no dissipation:

$$
\begin{align*}
&u_t + uu_x + \nu u_y = -p_x, \\
&v_t + uv_x + \nu v_y = -p_y + \theta, \\
&u_x + v_y = 0, \\
&\theta_t + u\theta_x + v\theta_y = \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy}
\end{align*}
$$


**Theorem**

Let $\kappa_1 > 0$ and $\kappa_2 > 0$ be fixed. Let $(\vec{u}_0, \theta_0) \in H^s$ with $s > 2$. Then there exists a unique global solution $(\vec{u}, \theta)$ with

$$
\vec{u} \in C([0, \infty); H^s). \quad \theta \in C([0, \infty); H^s) \cap L^2_{loc}(0, \infty; H^{s+1})
$$
Horizontal dissipation only, $\nu_1 > 0$

The 2D Boussinesq equations with horizontal dissipation only

$$
\begin{align*}
  u_t + uu_x + \nu u_y &= -p_x + \nu_1 u_{xx}, \\
  \nu_t + uv_x + \nu v_y &= -p_y + \nu_1 v_{xx} + \theta, \\
  u_x + v_y &= 0, \\
  \theta_t + u\theta_x + v\theta_y &= 0
\end{align*}
$$

(4)

Global existence and uniqueness.


The major effort is devoted to obtaining global bounds. We have global bound for the $L^2$-norm:

$$
\|(u(t), v(t))\|_{L^2}^2 + 2\nu \int_0^t \|(u_x, v_x)\|_{L^2}^2 d\tau = (\|(u_0, v_0)\|_{L^2} + t\|\theta_0\|_{L^2})^2
$$

$$
\|\theta(t)\|_{L^q}^q + \kappa q(q-1) \int_0^t \|\theta_x\|_{L^2}^{\frac{q-2}{2}} \|\theta\|_{L^2}^2 d\tau = \|\theta_0\|_{L^q}^q.
$$
The vorticity $\omega = \nabla \times u$ satisfies

$$\omega_t + u \cdot \nabla \omega = \nu \partial_{xx} \omega + \partial_x \theta$$

Since the partial derivative $\partial_{xx} \omega$ matches that of $\partial_x \theta$, the derivative in $\partial_x \theta$ can be shifted to $\omega$ through integration by parts in the process of energy estimates. Therefore, one can avoid bounding $\partial_x \theta$ and still get a global bound for $\omega$.

$$\frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 + \nu \| \partial_x \omega \|_{L^2}^2 = \int \partial_x \theta \omega = -\int \theta \partial_x \omega \leq \frac{\nu}{2} \| \partial_x \omega \|_{L^2}^2 + C(\nu) \| \theta \|_{L^2}^2$$
The 2D Boussinesq equations with horizontal dissipation only

\begin{equation*}
\begin{aligned}
& u_t + uu_x + vu_y = -p_x \\
& v_t + uv_x + vv_y = -p_y + \theta, \\
& u_x + v_y = 0, \\
& \theta_t + u\theta_x + v\theta_y = \kappa_1 \theta_{xx}
\end{aligned}
\end{equation*}

Vertical dissipation and thermal diffusion

Consider the 2D Boussinesq equations with vertical dissipation and thermal diffusion

\[
\begin{align*}
&u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\
&\nu_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta, \\
&u_x + v_y = 0, \\
&\theta_t + u\theta_x + v\theta_y = \kappa \theta_{yy}, \\
&u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \\
&\theta(x, y, 0) = \theta_0(x, y).
\end{align*}
\]
References


The major effort is devoted to obtaining global bounds. We have a global bound for the $L^2$-norm:

$$\|(u(t), v(t))\|_{L^2}^2 + 2\nu \int_0^t \|(u_y, v_y)\|_{L^2}^2 d\tau = (\|(u_0, v_0)\|_{L^2} + t\|\theta_0\|_{L^2})^2$$

$$\|\theta(t)\|_{L^q}^q + \kappa q(q - 1) \int_0^t \|\theta_y \theta\|_{L^2}^{q-2} \|\theta\|_{L^2}^2 d\tau = \|\theta_0\|_{L^q}^q.$$
The vorticity equation is

\[ \omega_t + u \omega_x + v \omega_y = \nu \omega_{yy} + \theta_x \]

and the mismatch of the derivatives in \( \omega_{yy} \) and \( \theta_x \) makes it much harder to derive a global bound for the vorticity. Therefore, it appears to be necessary to estimate \( \omega \) (or \( (\nabla u, \nabla v) \)) and \( \nabla \theta \) simultaneously.

\[
\frac{1}{2} \frac{d}{dt} \int \omega^2 + \nu \int (\partial_y \omega)^2 = \int \partial_x \theta \omega.
\]

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 + \kappa \int |\partial_y \nabla \theta|^2 = - \int \nabla \theta \cdot \nabla u \cdot \nabla \theta. \tag{6}
\]
To make use of the dissipation in the $y$-direction, we write

$$\nabla \theta \cdot \nabla u \cdot \nabla \theta = \partial_x u (\partial_x \theta)^2 + \partial_x v \partial_x \theta \partial_y \theta + \partial_y u \partial_x \theta \partial_y \theta + \partial_y v (\partial_y \theta)^2.$$ 

We bound the terms on the right.

$$\int \partial_x v \partial_x \theta \partial_y \theta = - \int \theta (\partial_x v \partial_{xy} \theta + \partial_{xy} v \partial_x \theta)$$

$$\leq \|\theta_0\|_{\infty} \|\partial_x v\|_2 \|\partial_{xy} \theta\|_2 + \|\theta_0\|_{\infty} \|\partial_x \theta\|_2 \|\partial_{xy} v\|_2$$

$$\leq \frac{\kappa}{4} \|\partial_{xy} \theta\|_2^2 + \frac{\nu}{4} \|\partial_{xy} v\|_2^2 + C \|\theta_0\|_{\infty}^2 (\|\partial_x v\|_2^2 + \|\partial_x \theta\|_2^2)$$

To bound the last two terms on the right, we need a lemma.
Lemma

Assume that $f, g, g_y, h, h_x \in L^2(\mathbb{R}^2)$. Then

$$\int \int |f g h| \, dx dy \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|g_y\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}}. \quad (7)$$

$$\left| \int \partial_y u \partial_x \theta \partial_y \theta \right| \leq C \|\partial_y u\|_2 \|\partial_x \theta\|_2^{\frac{1}{2}} \|\partial_y \theta\|_2^{\frac{1}{2}} \|\partial_x \theta\|_2^{\frac{1}{2}} \|\partial_y \theta\|_2^{\frac{1}{2}} \|\partial_{xy} \theta\|_2^{\frac{1}{2}}$$

$$\leq \frac{\kappa}{4} \|\partial_{xy} \theta\|_2^{\frac{2}{2}} + C(\kappa) \|\partial_y u\|_2^{\frac{2}{2}} \|\partial_x \theta\|_2^{\frac{2}{2}} \|\partial_y \theta\|_2^{\frac{2}{2}}.$$
However, the term $\int \partial_x u (\partial_x \theta)^2$ can NOT be bounded suitably.

But if we know

$$\int_0^T \| v(t) \|_{L^\infty}^2 \, dt < \infty, \quad (8)$$

then we have, after integration by parts,

$$\int \partial_x u (\partial_x \theta)^2 = - \int \partial_y v (\partial_x \theta)^2 = \int v \partial_x \theta \partial_{xy} \theta \leq \frac{\kappa}{4} \| \partial_{xy} \theta \|_2^2 + C(\kappa) \| v(t) \|_{L^\infty}^2 \| \partial_x \theta \|_2^2.$$

We are able to conclude that, if (8) holds, then

$$\| \omega \|_2^2 + \| \nabla \theta \|_2^2 + \nu \int (\partial_y \omega)^2 + \kappa \int |\partial_y \nabla \theta|^2 \leq C(T).$$

Unfortunately it appears to be extremely hard to prove (8).

Therefore, we have to solve this problem through a different route.
Consider the IVP for the anisotropic Boussinesq equations with vertical dissipation (5). Let $\nu > 0$ and $\kappa > 0$. Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$. Then, for any $T > 0$, (5) has a unique classical solution $(u, v, \theta)$ on $[0, T]$ satisfying

$$(u, v, \theta) \in C([0, T]; H^2(\mathbb{R}^2)).$$
The proof of the theorem relies on several ingredients.

**Proposition**

Assume \((u_0, v_0, \theta_0) \in H^2\). Let \((u, v, \theta)\) be the corresponding classical solution of (5). Then the quantity

\[
Y(t) = \|\omega\|^2_{H^1} + \|\theta\|^2_{H^2} + \|\omega^2 + |\nabla\theta|^2\|^2_2
\]

satisfies

\[
\frac{d}{dt} Y(t) + \|\omega_y\|^2_{H^1} + \|\theta_y\|^2_{H^2} + \int (\omega^2 + |\nabla\theta|^2) (\omega^2 + |\nabla\theta_y|^2) \\
\leq C \left(1 + \|\theta_0\|^2_\infty + \|v\|^2_\infty + \|u_y\|^2_2 + (1 + \|u\|^2_2) \|v_y\|^2_2\right) Y(t),
\]

where \(C\) is a constant.
Proposition

Let \((u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)\) and let \((u, v, \theta)\) be the corresponding classical solution of (5). Then,

\[
\sup_{r \geq 2} \frac{\|v(t)\|_{L^{2r}}}{{\sqrt{r \log r}}} \leq \sup_{r \geq 2} \frac{\|v_0\|_{L^{2r}}}{{\sqrt{r \log r}}} + B(t), \tag{9}
\]

where \(B(t)\) is an explicit integrable function of \(t \in [0, \infty)\) that depends on \(\nu, \kappa\) and the initial norm \(\|(u_0, v_0, \theta_0)\|_{H^2}\).
Proposition

Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$. Then there exists a constant $C$ depending on $s$ only such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} \left[ \log(e + \|f\|_{H^s(\mathbb{R}^2)}) \log \log(e + \|f\|_{H^s(\mathbb{R}^2)}) \right]^{\frac{1}{2}}.$$
Proof of Theorem 3.1: Applying Proposition 3.5 and using the simple fact that $\|v\|_{H^2}^2 \leq \|\omega\|_{H^1}^2 \leq Y(t)$, we obtain

$$\frac{d}{dt} Y(t) \leq A(t) Y(t) + C B^2(t) Y(t) \log(e + Y(t)) \log \log(e + Y(t)),$$

where $A(t) = C \left(1 + \|\theta_0\|_{\infty}^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2)\|v_y\|_2^2\right)$. An application of Gronwall’s inequality then concludes the proof of Theorem 3.1.
The proof for the following proposition

**Proposition**

Let \((u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)\) and let \((u, v, \theta)\) be the corresponding classical solution of (5). Then,

\[
\sup_{r \geq 2} \frac{\|v(t)\|_{L^2r}}{\sqrt{r \log r}} \leq \sup_{r \geq 2} \frac{\|v_0\|_{L^2r}}{\sqrt{r \log r}} + B(t),
\]

where \(B(t)\) is an explicit integrable function of \(t \in [0, \infty)\) that depends on \(\nu, \kappa\) and the initial norm \(\|(u_0, v_0, \theta_0)\|_{H^2}\).
Proof:

\[
\begin{aligned}
&\begin{cases}
    u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\
    v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta
\end{cases}
\end{aligned}
\]  

(11)

Taking the inner product of the second equation in (18) with \( v |v|^{2r-2} \) and integrating by parts, we obtain

\[
\frac{1}{2r} \frac{d}{dt} \int |v|^{2r} + \nu (2r - 1) \int v_y^2 |v|^{2r-2} 
= (2r - 1) \int p v_y |v|^{2r-2} + \int \theta v |v|^{2r-2}
\]

If we can bound the \( L^\infty \)-norm of \( p \), then we are pretty much done.
\[
(2r - 1) \int p v_y |v|^{2r-2} = (2r - 1) \int p |v|^{r-1} v_y |v|^{r-1}
\]
\[
\leq (2r - 1) \|p\|_\infty \|v|^{r-1}\|_2 \|v_y|v|^{r-1}\|_2
\]
\[
\leq \frac{\nu(2r - 1)}{4} \|v_y|v|^{r-1}\|_2^2 + C (2r - 1) \|p\|_\infty^2 \|v|^{2r-2}\|_2
\]
\[
\leq \frac{\nu(2r - 1)}{4} \|v_y|v|^{r-1}\|_2^2 + C (2r - 1) \|p\|_\infty^2 \|v|^{r-1}\|_2 \|v|^{2r-2-\frac{2}{r-1}}\|_2
\]

Then, if \(\int_0^T \|p\|_\infty^2 dt < \infty\),
\[
\frac{1}{2r} \frac{d}{dt} \|v\|_{2r}^{2r} \leq C (2r - 1) \|p\|_\infty^2 \|v|^{r-1}\|_2 \|v|^{2r-2-\frac{2}{r-1}}\|_2
\]

would yield \(\|v\|_{2r} \leq C \sqrt{r}\).
But unfortunately, we do not know if $\int_0^T \|p\|_\infty^2 \, dt < \infty$. What we can show is the following bound.

**Proposition**

Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ and let $(u, v, \theta)$ be the corresponding classical solution of (5). Then

$$\|(u(t), v(t))\|_4^4 + \nu \int_0^t \|(u_y(\tau), v_y(\tau))\|(u(\tau), v(\tau))\|_2^2 \, d\tau \leq M_1(t),$$

$$\|p(\cdot, t)\|_2, \|p(\cdot, t)\|_4 \leq M_2(t), \quad \int_0^t \|\nabla p(\cdot, \tau)\|_2^2 \, d\tau \leq M_3(t),$$

where $M_1, M_2$ and $M_3$ are explicit smooth functions of $t \in [0, \infty)$ that depend on $\nu, \kappa$ and the initial norm $\|(u_0, v_0, \theta_0)\|_{H^2}$.
We need several lemmas.

**Lemma**

Let $f \in H^1(\mathbb{R}^2)$. Let $R > 0$. Denote by $B(0, R)$ the ball centered at zero with radius $R$ and by $\chi_{B(0,R)}$ the characteristic function on $B(0, R)$. Write $f = \overline{f} + \tilde{f}$ with

$$\overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)} \mathcal{F} f) \quad \text{and} \quad \tilde{f} = \mathcal{F}^{-1}((1 - \chi_{B(0,R)}) \mathcal{F} f). \quad (12)$$

Then we have the following estimates for $\overline{f}$ and $\tilde{f}$.

(1) There exists a pure constant $C$ independent of $f$ and $R$ such that

$$||\overline{f}||_{L^\infty(\mathbb{R}^2)} \leq C \sqrt{\log R} ||f||_{H^1(\mathbb{R}^2)}. \quad (13)$$
Lemma

For any $2 \leq q < \infty$, there is a constant independent of $q$, $R$ and $f$ such that

$$\|\tilde{f}\|_{L^q(\mathbb{R}^2)} \leq C \frac{q}{R^q} \|\tilde{f}\|_{H^1(\mathbb{R}^2)} \leq C \frac{q}{R^q} \|f\|_{H^1(\mathbb{R}^2)}$$  \hspace{1cm} (14)

In particular, for $q = 4$,

$$\|\tilde{f}\|_{L^4(\mathbb{R}^2)} \leq \frac{C}{\sqrt{R}} \|f\|_{H^1(\mathbb{R}^2)}.$$
Lemma

Let $q \in [2, \infty)$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then

$$\int\int_{\mathbb{R}^2} |f \ g \ h| \, dx \, dy \leq C \|f\|_2 \|g\|_2^{\frac{1}{q}} \|g_y\|_2^{\frac{1}{q}} \|h\|_{2(q-1)}^{\frac{1}{q}} \|h_x\|_2^{\frac{1}{q}}. \quad (15)$$

where $C$ is a constant depending on $q$ only. Two special cases of (15) are

$$\int\int_{\mathbb{R}^2} |f \ g \ h| \, dx \, dy \leq C \|f\|_2 \|g\|_2^{\frac{3}{2}} ||g_y||_2^{\frac{3}{2}} ||h||_4^{\frac{3}{2}} ||h_x||_2^{\frac{3}{2}}. \quad (16)$$

and

$$\int\int_{\mathbb{R}^2} |f \ g \ h| \, dx \, dy \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} ||g_y||_2^{\frac{1}{2}} ||h||_2^{\frac{1}{2}} ||h_x||_2^{\frac{1}{2}}. \quad (17)$$
Proof of the global $L^{2r}$-bound:

\[
\begin{align*}
    u_t + uu_x + vu_y &= -p_x + \nu u_{yy}, \\
    v_t + uv_x + vv_y &= -p_y + \nu v_{yy} + \theta 
\end{align*}
\]

(18)

Taking the inner product of the second equation in (18) with $v |v|^{2r-2}$ and integrating by parts, we obtain

\[
\frac{1}{2r} \frac{d}{dt} \int |v|^{2r} + \nu (2r - 1) \int v_y^2 |v|^{2r-2} 
\]

\[
= (2r - 1) \int p v_y |v|^{2r-2} + \int \theta v |v|^{2r-2} 
\]

\[
= (2r - 1) \int \bar{p} v_y |v|^{2r-2} + (2r - 1) \int \tilde{p} v_y |v|^{2r-2} + \int \theta v |v|^{2r-2}.
\]
By Hölder’s inequality,

\[
\int \theta \, v \, |v|^{2r-2} \leq \|\theta\|_2 \|v\|_r^{2r-1},
\]

\[
\int \bar{p} \, v_y \, |v|^{2r-2} \leq \|\bar{p}\|_\infty \|v^{r-1}\|_2 \|v_y v^{r-1}\|_2.
\]

Applying Lemma 4, we have

\[
\int \tilde{p} \, v_y \, |v|^{2r-2} \leq C \|\tilde{p}\|_4^{\frac{2}{3}} \|\tilde{p}_x\|_2^{\frac{1}{3}} \|v^{r-1}\|_2^{\frac{2}{3}} \|(r-1)v_y v^{r-2}\|_2^{\frac{1}{3}} \|v_y v^{r-1}\|_2.
\]
Furthermore, by Hölder’s inequality,

\[ \|v^{r-1}\|_2 = \|v\|^{r-1}_{2(r-1)} \leq \|v\|_2^{\frac{1}{r-1}} \|v\|_2^{\frac{r(r-2)}{r-1}} , \]

\[ \|v^{r-2} v_y\|_2^2 = \int |v|^{2(r-2)} v_y^2 = \int |v|^{2(r-2)} v_y \frac{2(r-2)}{r-1} v_y \frac{2}{r-1} \leq \|v_y\|^{\frac{2}{r-1}}_2 \|v_y v^{r-1}\|_2. \]

Therefore,

\[ \int \bar{p} v_y |v|^{2r-2} \leq C \|\bar{p}\|_\infty \|v\|_2^{\frac{1}{2}} \|v\|_2^{\frac{r(r-2)}{r-1}} \|v_y v^{r-1}\|_2, \]

\[ \int \tilde{p} v_y |v|^{2r-2} \leq C (r-1)^{\frac{1}{3}} \|\tilde{p}\|_4^{\frac{2}{3}} \|\tilde{p}_x\|_2^{\frac{1}{3}} \|v\|_2^{\frac{2}{3(r-1)}} \|v\|_2^{\frac{2r(r-2)}{3(r-1)}} \]

\[ \times \|v_y\|_2^{\frac{1}{3(r-1)}} \|v_y v^{r-1}\|_2^{1+\frac{(r-2)}{3(r-1)}}. \]
By Young’s inequality and Lemma 2,

\[(2r-1) \int \tilde{p} v_y |v|^{2r-2} \leq \frac{\nu}{4} (2r-1) \| v y v^{r-1} \|_2^2 + C (2r-1)(\log R) \| p \|_{H^1} \| v \|_2^{\frac{2}{r-1}} \| v \|_{2r}^{2r-2-\frac{2}{r-1}}.\]

By Young’s inequality and Lemmas 2,

\[(2r-1) \int \tilde{p} v_y |v|^{2r-2} \leq \frac{\nu}{4} (2r-1) \| v y v^{r-1} \|_2^2 + C (2r-1)(r-1) \frac{2r-2}{2r-1} \frac{4}{2r-1} \| \tilde{p} \|_{L^4} \| \tilde{p}_x \|_2 \| v \|_2^{\frac{2}{r-1}} \| v \|_2^{\frac{4}{2r-1}} \| v \|_{2r}^{2r-2-\frac{2}{2r-1}} \frac{2r-2}{2r-1} \| \tilde{p} \|_{H^1} \| v \|_2^{\frac{2r-4}{2r-1}} \| v \|_2^{\frac{4}{2r-1}} \| v \|_{2r}^{2r-3-\frac{3}{2r-1}}.\]
Without loss of generality, we assume $\|v\|_{2r} \geq 1$. Inserting (19), (20) and (20) in (19), we have

$$\frac{1}{2r} \frac{d}{dt} \|v\|_{L^{2r}}^{2r} + \frac{\nu}{2} (2r - 1) \int v_y^2 |v|^{2r-2} \, dx$$

$$\leq C (2r - 1)(\log R) \|p\|^2_{H^1} \|v\|^\frac{2}{r-1} \|v\|^{2r-2}_{2r}$$

$$+ C (2r - 1)(r - 1)^\frac{2r-2}{2r-1} R^{-\frac{r-1}{2r-1}} \|p\|^\frac{2r-2}{2r-1} \|p\|_{H^1}^{\frac{4r-4}{2r-1}} \|v_y\|^\frac{2}{2r-1} \|v\|^{\frac{4}{2r-1}} \|v\|^{2r-2}_{2r}$$

$$+ \|\theta\|_{L^2} \|v\|^{2r-1}_{L^{2r}}.$$
Especially,

\[
\frac{d}{dt} \| v \|_{L^{2r}}^2 \leq C(2r - 1)(\log R) \| p \|_{H^1}^2 \| v \|_2^{2r-1} \\
+ C (2r - 1)(r - 1)^{\frac{2r-2}{2r-1}} R^{-\frac{r-1}{2r-1}} \| p \|_{L^4}^{2r-2} \| p \|_{H^1}^{2r-1} \left( \| p \|_{H^1}^2 + \| v_y \|_2^2 \right) \| v \|_2^{4} \\
+ \| \theta \|_{L^{2r}}^2 + \| v \|_{L^{2r}}^2.
\]

Taking \( R = (2r - 1)^{\frac{2r-1}{2r-2}} (r - 1)^2 \), integrating in time and applying Propositions, we obtain
||v(t)||_{L^2}^{2r} \leq ||v_0||_{L^2}^{2r} + B_1(t)r \log r + B_2(t),

where $B_1$ and $B_2$ are explicit integrable functions. Therefore,

$$\sup_{r \geq 2} \frac{||v(t)||_{L^2}^{2r}}{r \log r} \leq \sup_{r \geq 2} \frac{||v_0||_{L^2}^{2r}}{r \log r} + (B_1(t) + B_2(t)).$$

This completes the proof of Proposition 3.5.
Consider the 2D fractional Boussinesq equations

\[
\begin{align*}
\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} + \nu \Lambda^\alpha \mathbf{u} &= -\nabla p + \theta \mathbf{k}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\theta_t + \mathbf{u} \cdot \nabla \theta + \kappa \Lambda^\beta \theta &= 0
\end{align*}
\]

where \( \Lambda = (-\Delta)^{\frac{1}{2}} \). When \( \alpha = \beta = 2 \), these equations reduce to the standard 2D Boussinesq.
Three cases:

The subcritical case: $\alpha + \beta > 1$;

The critical case: $\alpha + \beta = 1$;

The supercritical case: $\alpha + \beta < 1$.

Energy estimates does not yield any global bounds on the Sobolev norms unless $\alpha + \beta \geq 2$. 
The global here is to get the global regularity for the smallest $\alpha$ and $\beta$. As we know, the generalized 3D Navier-Stokes equations

$$\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + (-\Delta)^\alpha \tilde{u} = -\nabla p$$

$$\nabla \cdot \tilde{u} = 0$$

has global regularity for $\alpha \geq \frac{5}{4}$. 
Global regularity in the case $\nu > 0$, $\kappa = 0$ and $\alpha = 1$, namely

$$
\begin{cases}
\overrightarrow{u}_t + \overrightarrow{u} \cdot \nabla \overrightarrow{u} + \Lambda \overrightarrow{u} = -\nabla p + \theta \overrightarrow{k}, \\
\nabla \cdot \overrightarrow{u} = 0, \\
\theta_t + \overrightarrow{u} \cdot \nabla \theta = 0
\end{cases}
$$

Hmidi, Keraani and Rousset, JDE, 2010. Also in arXiv: 0904.1536

v1 [math.AP]
Theorem

Let $u_0 \in H^1 \cap \dot{W}^{1,q}$ with $q \in (2, \infty)$. Let $\theta_0 \in L^2 \cap B^0_{\infty,1}$. Then the Boussinesq equations have a unique global solution satisfying

$$u \in L^\infty_{loc}([0, \infty); H^1 \cap \dot{W}^{1,q}) \cap L^1_{loc}([0, \infty); B^1_{\infty,1}),$$

$$\theta \in L^\infty_{loc}([0, \infty); L^2 \cap B^0_{\infty,1}).$$
Global regularity in the case $\nu = 0$, $\kappa > 0$ and $\beta = 1$, namely

$$\begin{cases} \overrightarrow{u}_t + \overrightarrow{u} \cdot \nabla \overrightarrow{u} = -\nabla p + \theta \overrightarrow{k}, \\
\nabla \cdot \overrightarrow{u} = 0, \\
\theta_t + \overrightarrow{u} \cdot \nabla \theta + \Lambda \theta = 0 \end{cases}$$

Hmidi, Keraani and Rousset, CPDE, 2011, arXiv: 0903.3747 v1

[math.AP]
Theorem

Let $u_0 \in B^{1}_{\infty,1} \cap \dot{W}^{1,q}$ with $q \in (2, \infty)$. Let $\theta_0 \in L^q \cap B^0_{\infty,1}$. Then the Boussinesq equations have a unique global solution satisfying

$$u \in L^\infty_{loc}([0, \infty); B^{1}_{\infty,1} \cap \dot{W}^{1,q}),$$

$$\theta \in L^\infty_{loc}([0, \infty); L^q \cap B^0_{\infty,1}) \cap \tilde{L}^1_{loc}([0, \infty); B^1_{q,\infty}).$$

**Theorem**

Let $\alpha \in ((6 - \sqrt{6})/4, 1)$ and $\beta \in (1 - \alpha, \min ((7 + 2\sqrt{6})\alpha/5 - 2, \alpha(1 - \alpha)/(\sqrt{6} - 2\alpha), 2 - 2\alpha))$.

Let $u_0 \in H^1 \cap W^{1,q}$ with $q \in (1/(\alpha + \beta - 1), \infty)$ and $\theta_0 \in H^{1-\alpha/2} \cap B^{1-\alpha/2}_{\infty,1}$. Then the fractional Boussinesq equations have a unique global solution.

Note that in this theorem $\alpha + \beta > 1$, a subcritical case.

Global regularity for the subcritical case by a different approach

**Theorem**

Assume that \((u_0, \theta_0) \in S\), the Schwartz class. If \(\beta > \frac{2}{2 + \alpha}\), then the Boussinesq equations has a unique global smooth solution.

The proof involves the pointwise inequality for fractional Laplacian

\[
\nabla f \cdot \Lambda^\alpha \nabla f(x) \geq \frac{1}{2} \Lambda^\alpha |\nabla f|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c \|f\|_{L^\infty}^\alpha}
\]

Note that in this theorem \(\alpha + \beta > 1\), a subcritical case.
In a recent preprint we obtained the global regularity for the general critical dissipation $\alpha + \beta = 1$.

Theorem

Let $\alpha_0 < \alpha < 1$ and $\alpha + \beta = 1$, where

$$\alpha_0 = \frac{23 - \sqrt{145}}{12} \approx 0.9132. \quad (20)$$

Assume that $u_0 \in B_{2,1}^\sigma(\mathbb{R}^2)$ with $\sigma \geq \frac{5}{2}$ and $\theta_0 \in B_{2,1}^{2}(\mathbb{R}^2)$. Then the fractional Boussinesq has a unique global solution $(u, \theta)$ satisfying, for any $0 < T < \infty$,

$$u \in C([0, T]; B_{2,1}^\sigma(\mathbb{R}^2)) \cap L^1([0, T]; B_{2,1}^{\sigma+\alpha}(\mathbb{R}^2)),$$

$$\theta \in C([0, T]; B_{2,1}^{2}(\mathbb{R}^2)) \cap L^1([0, T]; B_{2,1}^{2+\beta}(\mathbb{R}^2)). \quad (21)$$
Direct energy estimates would not work.

\[ \partial_t \omega + u \cdot \nabla \omega + \Lambda^\alpha \omega = \partial_1 \theta. \]

Multiplying by \( \omega \) and integrating in space yields

\[ \frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 + \| \Lambda^{\frac{\alpha}{2}} \omega \|_{L^2}^2 = \int \partial_1 \theta \omega \]

Since we have no control on \( \partial_1 \theta \), we integrate by parts

\[ \int \partial_1 \theta \omega = - \int \theta \partial_1 \omega \leq \| \theta \|_{L^2} \| \partial_1 \omega \|_{L^2} \]

Then, we need \( \frac{\alpha}{2} \geq 1 \), or \( \alpha \geq 2 \)

\[ \frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 + \| \Lambda^{\frac{\alpha}{2}} \omega \|_{L^2}^2 \leq \| \theta \|_{L^2} \| \partial_1 \omega \|_{L^2} \]
Proof. Some key steps:

First, we resort to the vorticity equation \( \omega = \nabla \times u \),

\[
\begin{aligned}
\partial_t \omega + u \cdot \nabla \omega + \Lambda^\alpha \omega &= \partial_1 \theta, \\
\langle u \rangle = \nabla^\perp \psi, \quad \Delta \psi &= \omega \quad \text{or} \quad u = \nabla^\perp (-\Delta)^{-1} \omega.
\end{aligned}
\]

(22)

“\( \partial_1 \theta \)” is the “vortex stretching” term. Writing

\( \Lambda^\alpha \omega - \partial_1 \theta = \Lambda^\alpha (\omega - \Lambda^{-\alpha} \partial_1 \theta) \), we can hide \( \partial_1 \theta \) by considering

\[
G = \omega - R_\alpha \theta \quad \text{with} \quad R_\alpha = \Lambda^{-\alpha} \partial_1,
\]
Fluid PDEs

2D Boussinesq with fractional dissipation

General critical case

\[ \partial_t \omega + u \cdot \nabla \omega + \Lambda^\alpha G = 0 \]

\[ \partial_t R_\alpha \theta + u \cdot \nabla R_\alpha \theta + \Lambda^{\beta - \alpha} \partial_1 \theta = -[R_\alpha, u \cdot \nabla] \theta \]

We get, by taking their differences,

\[ \partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [R_\alpha, u \cdot \nabla] \theta + \Lambda^{\beta - \alpha} \partial_1 \theta. \quad (23) \]

Here we have used the standard commutator notation

\[ [R_\alpha, u \cdot \nabla] \theta = R_\alpha (u \cdot \nabla \theta) - u \cdot \nabla R_\alpha \theta. \]

The commutator term \([R_\alpha, u \cdot \nabla] \theta\) is less singular than \(\partial_1 \theta\) in the vorticity equation.
For $\alpha > \alpha_0$, $G$ is regular, namely $\| \nabla \nabla \perp (-\Delta)^{-1} G \|_{L^\infty} < \infty$.

$$G = \omega - \mathcal{R}_\alpha \theta = \omega - \Lambda^{-\alpha} \partial_1 \theta.$$ 

By the Biot-Savart law, $u = \nabla \perp (-\Delta)^{-1} \omega$,

$$u = \nabla \perp (-\Delta)^{-1} \omega = \nabla \perp (-\Delta)^{-1} G + \nabla \perp (-\Delta)^{-1} \Lambda^{-\alpha} \partial_1 \theta \equiv \tilde{u} + v$$

For $\alpha + \beta = 1$, we deduce the generalized critical SQG

$$\begin{align*}
\begin{cases}
\partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\
u = \tilde{u} + v, \quad v = \nabla \perp \Lambda^{-3} \Lambda^\beta \partial_1 \theta,
\end{cases}
\end{align*}$$

$$\Lambda^{1-\beta} v = \nabla \perp \Lambda^{-2} \partial_1 \theta.$$
The surface quasi-geostrophic (SQG) equation has been studied by many researchers and significant progress has been made on the global regularity issue. The critical SQG problem has been resolved and there are four proofs (Kiselev-Nazarov-Volberg, Caffarelli-Vasseur, Kiselev-Nazarov and Constantin-Vicol). The SQG has been studied by many others.

The generalized SQG has been studied by Constantin-Iyer-Wu, Kiselev, etc. The global regularity of this SQG can then be established by modifying the approach of Constantin-Vicol.
Results beyond the critical case

The global well-posedness for the supercritical case $\alpha + \beta < 1$ remains open:

$$
\begin{aligned}
\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nu \Lambda^\alpha \vec{u} &= -\nabla p + \theta \vec{k}, \\
\nabla \cdot \vec{u} &= 0, \\
\theta_t + \vec{u} \cdot \nabla \theta + \kappa \Lambda^\beta \theta &= 0
\end{aligned}
$$

where $\Lambda = (\Delta)^{1/2}$. Eventual regularity type result and global regularity for slightly more singular velocity can be established.
Q. Jiu, J. Wu and W. Yang, Eventual regularity of the 2D Boussinesq equations with supercritical dissipation, preprint.

**Theorem**

Consider the supercritical Boussinesq with \( \alpha > \alpha_0, \beta > 0 \) and \( \alpha + \beta < 1 \), where

\[
\alpha_0 = \frac{23 - \sqrt{145}}{12} \approx 0.9132. \tag{24}
\]

Assume that \((u_0, \theta_0) \in H^s(\mathbb{T}^2)\) with \(s > 2\), and \(u_0\) and \(\theta_0\) have zero mean. Let \((u, \theta)\) be a global weak solution. Then, there exist \(0 < T_1 \leq T_2 < \infty\) such that \((u, \theta)\) is actually a classical solution on \([0, T_1]\) and on \([T_2, \infty)\).
2D Incompressible MHD

The standard 2D MHD equations can be written as

\[
\begin{align*}
    u_t + u \cdot \nabla u & = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b & = \eta \Delta b + b \cdot \nabla u, \\
    \nabla \cdot u & = 0, \quad \nabla \cdot b = 0,
\end{align*}
\]

where \( u \) denotes the velocity field, \( b \) the magnetic field, \( p \) the pressure, \( \nu \) the viscosity and \( \eta \) the magnetic diffusivity.
The MHD equations model electrically conducting fluid in the presence of a magnetic field. The first equation is the Navier-Stokes equation with the Lorentz force generated by the magnetic field and the second equation is the induction equation for the magnetic field.

The MHD equations model many phenomena such as the geomagnetic dynamo in geophysics and solar winds and solar flares in astrophysics. The MHD is also used in dealing with engineering problems such as plasma confinement.
Mathematically the 2D MHD equations may serve as a lower-dimensional model of the 3D hydrodynamics equations. The main difficulty in dealing with the 2D MHD is due to the strong nonlinear coupling.
We consider the initial-value problems of the MHD equations with the initial data

\[ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x). \]

What we care about is the global regularity issue: Do these IVPs have a global solution for sufficiently smooth data \((u_0, b_0)\)?

The global regularity problem on the 2D MHD equations has attracted considerable attention recently from the PDE community and progress has been made.
Ideal MHD equations:

\[
\begin{aligned}
&u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\
&b_t + u \cdot \nabla b = b \cdot \nabla u, \\
&\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\
&u(x, y, 0) = u_0(x, y), \quad b(x, y, 0) = b_0(x, y).
\end{aligned}
\]  

(26)

The global regularity problem remains open, although we do have local well-posedness and regularity criteria.
Theorem

Given \((u_0, b_0) \in H^s(\mathbb{R}^2)\) with \(s > 2\). Then there exists a unique local classical solution \((u, b) \in C([0, T_0); H^s)\) for some \(T_0 > 0\). In addition, if

\[
\int_0^T (\|\omega\|_\infty + \|j\|_\infty) \, dt < \infty
\]

for \(T > T_0\), then the solution remains in \(H^s\) for any \(t \leq T\).

Why is the global regularity problem hard? The global \(L^2\)-bound for \((u, b)\) follows directly from the MHD equations

\[
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\]
But global bounds for any Sobolev-norm appear to be impossible, for example, the $H^1$-norm. Consider the equations of $\omega = \nabla \times u$ and $j = \nabla \times b$,

$$\begin{align*}
\omega_t + u \cdot \nabla \omega &= b \cdot \nabla j, \\
j_t + u \cdot \nabla j &= b \cdot \nabla \omega + 2\partial_x b_1 (\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1 (\partial_y b_1 + \partial_x b_2)
\end{align*}$$
Clearly,

\[ \frac{1}{2} \frac{d}{dt} \left( \| \omega \|_{L^2}^2 + \| j \|_{L^2}^2 \right) = 2 \int j \partial_x b_1 \partial_y u_1 + \cdots \]

Since there is no dissipation, we need

\[ \int_0^T \| \omega \|_\infty \, dt < \infty \quad \text{or} \quad \int_0^T \| j \|_\infty \, dt < \infty \]

in order for this differential inequality to be closed. To bound Sobolev norms with more than one derivative, we need both. In fact, for \( Y = \| u(t) \|_{H^s}^2 + \| b \|_{H^s}^2 \),

\[ \frac{d}{dt} Y(t) \leq C \left( \| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty} \right) Y(t). \]
Dissipative MHD equations

Fully dissipative MHD equations:

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + \nu \Delta u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= \eta \Delta b + b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0.
\end{align*}
\]

The global regularity can be easily established.

Theorem

Let \((u_0, b_0) \in H^1(\mathbb{R}^2)\). Then there exists a unique global strong solution \((u, b)\) satisfying, for any \(T > 0\),

\[
    u, b \in L^\infty([0, T]; H^1(\mathbb{R}^2)) \cap L^2([0, T]; H^2(\mathbb{R}^2))
\]
The global regularity problem for the ideal MHD equations is extremely difficult while the problem for the fully dissipative MHD equations is very easy. Naturally we would like to consider the cases with intermediate dissipation.
The 2D MHD equations with no magnetic diffusion:

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + \nu \Delta u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0,
\end{align*}
\]

where $\nu > 0$. The global regularity problem remains open. Again we have the global $L^2$-bound

\[
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\]
The dissipation does not appear to be enough to obtain global bounds in Sobolev spaces:

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= \nu \Delta \omega + b \cdot \nabla j, \\
j_t + u \cdot \nabla j &= b \cdot \nabla \omega + 2 \partial_x b_1 (\partial_y u_1 + \partial_x u_2) - 2 \partial_x u_1 (\partial_y b_1 + \partial_x b_2).
\end{align*}
\]

\[
\frac{1}{2} \frac{d}{dt} (\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \nu \|\nabla \omega\|_{L^2}^2 = 2 \int j \partial_x b_1 \partial_y u_1 + \cdots
\]

To close the inequality, we need

\[
\int_0^T \|\nabla u\|_{L^\infty} \, dt < \infty \quad \text{or} \quad \int_0^T \|\omega\|_{L^\infty} \, dt < \infty.
\]
This simple calculation also shows that the system

\[
\begin{cases}
u_t + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^\alpha u + b \cdot \nabla b, \\b_t + u \cdot \nabla b = b \cdot \nabla u, \\
\nabla \cdot u = 0, \quad \nabla \cdot b = 0
\end{cases}
\]

(27)

with \( \nu > 0 \) and \( \alpha \geq 2 \) does have a global solution since the

\( L^2 \)-estimate already provides

\[
\int_0^T \| \omega(t) \|^2_{H^1} \, dt < \infty,
\]

which is more or less \( \int_0^T \| \omega \|_\infty \, dt < \infty \).
In fact, Yamazaki was able to show a global regularity result for

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p - \nu \frac{(-\Delta)^2}{\log(3-\Delta)} u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0
\end{align*}
\]

(28)

Z. Lei also has an interesting work on the 3D axisymmetric MHD equations with no magnetic diffusion:

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + \nu \Delta u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \\
    \nabla \cdot b &= 0,
\end{align*}
\]

Theorem

Assume that $u_0$ and $b_0$ are axisymmetric divergence-free vector fields with

$$u_0^\theta = 0, \quad b_0^r = b_0^z = 0,$$

and $u_0 \in H^s$, $b_0 \in H^s$ with $s \geq 2$, and $b_0^\theta/r \in L^\infty$.

Then the 3D MHD equations with no magnetic diffusion have a global solution satisfying

$$\|u(t)\|^2_{H^2} + \|b(t)\|^2_{H^2} + \|\nabla u\|^2_{L_t^2 H^2} \leq \exp(\exp(\exp t^{5/4})).$$
Global small solutions

Progress has been made on small global solutions for

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + \nu \Delta u + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0.
\end{align*}
\]

Global solutions near an equilibrium state have been obtained in

Since $\nabla \cdot b = 0$, write $b = \nabla \perp \phi$ and

$$
\left\{ \begin{array}{l}
  u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + \nabla \perp \phi \cdot \nabla \nabla \perp \phi, \\
  \phi_t + u \cdot \nabla \phi = 0, \\
  \nabla \cdot u = 0.
\end{array} \right.
$$

Clearly, $u \equiv 0$ and $\phi = y$ is a steady solution. Setting $\phi = y + \psi$ yields

$$
\left\{ \begin{array}{l}
  \partial_t \psi + u \cdot \nabla \psi + u_2 = 0, \\
  \partial_t u_1 + u \cdot \nabla u_1 - \nu \Delta u_1 + \partial_1 \partial_2 \psi = -\partial_1 p - \nabla \cdot (\partial_1 \psi \nabla \psi), \\
  \partial_t u_2 + u \cdot \nabla u_2 - \nu \Delta u_2 + \partial_1^2 \psi = -\partial_2 p - \nabla \cdot (\partial_2 \psi \nabla \psi),
\end{array} \right.
$$
They reformulated the system in Lagrangian coordinates. More precisely, they define

$$Y(x, y, t) = X(x, y, t) - (x, y),$$

where $X = X(x, y, t)$ be the particle trajectory determined by $u$. $Y$ satisfies

$$Y_{tt} - \Delta Y_t - \partial_x^2 Y = f(Y, q(y)), \quad q = p + |\nabla \psi|^2.$$

They then estimate the Lagrangian velocity $Y_t$ in $L^1_tLip_x$, using anisotropic Littlewood-Paley theory and anisotropic Besov space techniques.
The structure of the linear equation

\[ Y_{tt} - \Delta Y_t - \partial_x^2 Y = 0 \]

played a crucial role. The characteristic equation satisfies

\[ \lambda^2 + |\xi|^2 \lambda + \xi_1^2 = 0, \]

which has two roots

\[ \lambda_\pm = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_1^2}}{2}. \]

As \(|\xi| \to \infty\), \(\lambda_+ \to -|\xi|^2\) and \(\lambda_- \to -1\).
Theorem

Let $s_1 > 1$, $s_2 \in (-1, -\frac{1}{2})$ and $s > s_1 + 2$. Given $\psi_0$ and $u_0$ satisfying $$(\nabla \psi_0, u_0) \in H^s \cap \dot{H}^{s_2},$$ and

$$\|\nabla \psi_0\|_{H^{s_1+2}} \leq 1, \quad \| (\nabla \psi_0, u_0) \|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} + \| \partial_{x_2} \psi_0 \|_{H^{s_1+2}} \leq \epsilon_0$$

for some $\epsilon_0$ small. Assume that $\partial_{x_2} \psi_0$ and $(1 + \partial_{x_2} \psi_0, -\partial_{x_1} \psi_0)$ are admissible on $0 \times \mathbb{R}$ and $\partial_{x_2} \psi_0(\cdot, x_2) \subset [-K, K]$ for some $K$. Then the 2D MHD equations with no magnetic diffusion has a unique global solution $(\psi, u, p)$. 
We call that $f$ and $b$ are admissible on a domain $D$ of $\mathbb{R}^2$ if there holds

$$\int_{\mathbb{R}} f(X(a, t))dt = 0 \quad \text{for all } a \in D.$$  

where $X$ is the particle trajectory defined by $b$.  


Global small solutions for a damped system


Consider the following 2D MHD equation

\[
\begin{cases}
\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \vec{u} + \nabla P = -\text{div}(\nabla \phi \otimes \nabla \phi), & (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t \phi + \vec{u} \cdot \nabla \phi = 0, \\
\nabla \cdot \vec{u} = 0, \\
\vec{u}\big|_{t=1} = \vec{u}_0(x, y), \quad \phi\big|_{t=1} = \phi_0(x, y),
\end{cases}
\]

(29)

where \( \vec{u} = (u, v) \).
Letting $\phi = y + \psi$ in (29) yields

\[
\begin{cases}
\partial_t u + u \partial_x u + v \partial_y u + u + \partial_x \tilde{P} = -\Delta \psi \partial_x \psi, \\
\partial_t v + u \partial_x v + v \partial_y v + v + \partial_y \tilde{P} = -\Delta \psi - \Delta \psi \partial_y \psi, \\
\partial_t \psi + u \partial_x \psi + v \partial_y \psi + v = 0, \\
\partial_x u + \partial_y v = 0,
\end{cases}
\]

(30)

where $\tilde{P} = P + \frac{1}{2} |\nabla \phi|^2$. By $\nabla \cdot \vec{u} = 0$,

$$\Delta \tilde{P} = -\nabla \cdot (\vec{u} \cdot \nabla \vec{u}) - \nabla \cdot (\Delta \psi \nabla \psi) - \Delta \partial_y \psi.$$
Therefore, (30) can be written as

\[
\begin{align*}
\partial_t u + u - \partial_{xy} \psi &= N_1, \\
\partial_t v + v + \partial_{xx} \psi &= N_2, \\
\partial_t \psi + u \partial_x \psi + v \partial_y \psi + v &= 0,
\end{align*}
\]

(31) (32) (33)

where

\[
\begin{align*}
N_1 &= -\bar{u} \cdot \nabla u + \partial_x \Delta^{-1} \nabla \cdot (\bar{u} \cdot \nabla \bar{u}) - \Delta \psi \partial_x \psi + \partial_x \Delta^{-1} \nabla \cdot (\Delta \psi \nabla \psi), \\
N_2 &= -\bar{u} \cdot \nabla v + \partial_y \Delta^{-1} \nabla \cdot (\bar{u} \cdot \nabla \bar{u}) - \Delta \psi \partial_y \psi + \partial_y \Delta^{-1} \nabla \cdot (\Delta \psi \nabla \psi).
\end{align*}
\]
Taking the time derivative leads to

\[
\begin{aligned}
\partial_{tt} u + \partial_t u - \partial_{xx} u &= F_1, \\
\partial_{tt} v + \partial_t v - \partial_{xx} v &= F_2, \\
\partial_{tt} \psi + \partial_t \psi - \partial_{xx} \psi &= F_0,
\end{aligned}
\]

\[
\begin{array}{l}
\vec{u} \big|_{t=1} = \vec{u}_0(x, y), \quad \vec{u}_t \big|_{t=1} = \vec{u}_1(x, y) \\
\psi \big|_{t=1} = \psi_0(x, y), \quad \psi_t \big|_{t=1} = \psi_1(x, y),
\end{array}
\]

where \( \vec{u}_1 = (u_1(x, y), v_1(x, y)) \), \( \psi_0 = \phi_0 - y \), and

\[
\begin{aligned}
u_1 &= (-u + \partial_{xy} \psi + N_1) \big|_{t=1}, \\
\psi_1 &= (-u \partial_x \psi - v \partial_y \psi - v) \big|_{t=1},
\end{aligned}
\]
and

\[ F_0 = -\bar{u} \cdot \nabla \psi - \partial_t (\bar{u} \cdot \nabla \psi) - N_2, \]

\[ F_1 = \partial_t N_1 - \partial_{xy} (\bar{u} \cdot \nabla \psi), \]

\[ F_2 = \partial_t N_2 + \partial_{xx} (\bar{u} \cdot \nabla \psi). \]
We consider the linear equation

$$\partial_{tt} \Phi + \partial_t \Phi - \partial_{xx} \Phi = 0, \quad (35)$$

with the initial data

$$\Phi(0, x, y) = \Phi_0(x, y), \quad \Phi_t(0, x, y) = \Phi_1(x, y).$$

Taking the Fourier transform on the equation (35), we have

$$\partial_{tt} \hat{\Phi} + \partial_t \hat{\Phi} + \xi^2 \hat{\Phi} = 0, \quad (36)$$

where the Fourier transform $\hat{\Phi}$ is defined as

$$\hat{\Phi}(t, \xi, \eta) = \int_{\mathbb{R}^2} e^{ix\xi + iy\eta} \Phi(t, x, y) \, dx\, dy.$$
Solving (36) by a simple ODE theory, we have

\[
\hat{\Phi}(t, \xi, \eta) = \frac{1}{2} \left( e^{-\frac{1}{2} + \sqrt{\frac{1}{4} - \xi^2}} t + e^{-\frac{1}{2} - \sqrt{\frac{1}{4} - \xi^2}} t \right) \hat{\Phi}_0(\xi, \eta) \\
+ \frac{1}{2\sqrt{\frac{1}{4} - \xi^2}} \left( e^{-\frac{1}{2} + \sqrt{\frac{1}{4} - \xi^2}} t - e^{-\frac{1}{2} - \sqrt{\frac{1}{4} - \xi^2}} t \right) \\
\left( \frac{1}{2} \hat{\Phi}_0(\xi, \eta) + \hat{\Phi}_1(\xi, \eta) \right).
\]
Definition

Let the operators $K_0(t, \partial_x), K_1(t, \partial_x)$ be defined as

$$K_0(t, \partial_x)f(t, \xi, \eta) = \frac{1}{2} \left( e^{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \xi^2}\right)t} + e^{\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - \xi^2}\right)t} \right) \hat{f}(t, \xi, \eta);$$

and

$$K_1(t, \partial_x)f(t, \xi, \eta) = \frac{1}{2 \sqrt{\frac{1}{4} - \xi^2}} \left( e^{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \xi^2}\right)t} - e^{\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - \xi^2}\right)t} \right) \hat{f}(t, \xi, \eta).$$

where $\sqrt{-1} = i$. 

Therefore, the solution \( \Phi \) of the equation (35) is written as

\[
\Phi(t, x, y) = K_0(t, \partial_x)\Phi_0 + K_1(t, \partial_x)(\frac{1}{2}\Phi_0 + \Phi_1).
\]

Moreover, consider the inhomogeneous equation,

\[
\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = F, \tag{37}
\]

with initial data \( \Phi(1, x) = \Phi_0, \partial_t\Phi(1, x) = \Phi_1 \).
Then we have the following standard Duhamel formula,

\[
\Phi(t, x, y) = K_0(t, \partial_x)\Phi_0 + K_1(t, \partial_x)\left(\frac{1}{2}\Phi_0 + \Phi_1\right) 
+ \int_1^t K_1(t - s, \partial_x)F(s, x, y) \, ds.
\] (38)

The rest of the proof is to apply this formula to rewrite (34) and then verify the continuity principle. We will need the following estimates on $K_0$ and $K_1$. 
Lemma

Let $K_0, K_1$ be defined in Definition (5.5), then

1) $\| |\xi|^\alpha \hat{K}_i(t, \cdot) \|_{L^q_\xi(|\xi| \leq \frac{1}{2})} \lesssim t^{-\frac{1}{2}(\frac{1}{q} + \alpha)}$, for any $\alpha \geq 0$,

$1 \leq q \leq \infty$, $i = 0, 1$.

2) $\| \partial_t \hat{K}_i(t, \cdot) \|_{L^q_\xi(|\xi| \leq \frac{1}{2})} \lesssim t^{-1 - \frac{1}{2q}}$, $i = 0, 1$.

3) $|\hat{K}_i(t, \xi)| \lesssim e^{-\frac{1}{2}t}$, for any $|\xi| \geq \frac{1}{2}$, $i = 0, 1$.

4) $|\langle \xi \rangle^{-1} \partial_t \hat{K}_0(t, \xi)|, |\partial_t \hat{K}_1(t, \xi)| \lesssim e^{-\frac{1}{2}t}$, for any $|\xi| \geq \frac{1}{2}$.
Proposition

Let $K(t, \partial_x)$ be a Fourier multiplier operator satisfying

$$\|\hat{\partial_x^\alpha K(t, \xi)}\|_{L^1_\xi(|\xi| \leq \frac{1}{2})} < \infty, \quad \|\hat{K}(t, \xi)\|_{L^\infty_\xi(|\xi| \geq \frac{1}{2})} < \infty, \quad \alpha \geq 0.$$  

Then, for any space-time Schwartz function $f$,

$$\|\hat{\partial_x^\alpha K(t, \partial_x)} f\|_{L^\infty_{xy}} \lesssim \left(\|\hat{\partial_x^\alpha K(t, \xi)}\|_{L^1_\xi(|\xi| \leq \frac{1}{2})} + \|\hat{K}(t, \xi)\|_{L^\infty_\xi(|\xi| \geq \frac{1}{2})}\right)$$

$$\times \|\langle \nabla \rangle^{\alpha+1+\epsilon} \partial_y f\|_{L^1_{xy}}.$$  

(40)
Let $X_0$ be the Banach space defined by the following norm

\[
\| (\tilde{u}_0, \psi_0) \|_{X_0} = \| \langle \nabla \rangle^N (\tilde{u}_0, \nabla \psi_0) \|_{L^2_{xy}} + \| \langle \nabla \rangle^{6+} (\tilde{u}_0, \psi_0) \|_{L^1_{xy}} + \| \langle \nabla \rangle^{6+} (\tilde{u}_1, \psi_1) \|_{L^1_{xy}},
\]

where $\langle \nabla \rangle = (I - \Delta)^{\frac{1}{2}}$, $N \gg 1$ and $a^+ \text{ denotes } a + \epsilon$ for small $\epsilon > 0$.

The solution spaces $X$ is defined by

\[
\| (\tilde{u}, \psi) \|_X = \sup_{t \geq 1} \left\{ t^{-\epsilon} \| \langle \nabla \rangle^N (\tilde{u}(t), \nabla \psi(t)) \|_2 + t^{\frac{1}{4}} \| \langle \nabla \rangle^3 \psi \|_2 \\
+ t^{\frac{1}{4}} \| \langle \nabla \rangle^3 \psi \|_2 + t^{\frac{3}{2}} \| \partial_{xx} \psi \|_\infty + t^{\frac{5}{4}} \| \langle \nabla \rangle^2 \partial_{xx} \psi \|_2 + t^{\frac{3}{2}} \| \partial_{xxx} \psi \|_2 \\
+ t^{\frac{3}{2}} \| \partial_t \tilde{u} \|_\infty + t^{\frac{5}{4}} \| \langle \nabla \rangle \partial_t \tilde{u} \|_2 + t \| \langle \nabla \rangle \partial_x \tilde{u} \|_\infty + t^{\frac{3}{2}} \| \partial_x \partial_t v \|_2 \right\}.
\]
Our main result can be stated as follows:

**Theorem**

Let \( \psi_0 = \phi_0 - y \) and \( \psi = \phi - y \). Then there exists a small constant \( \varepsilon > 0 \) such that, if the initial data \((\vec{u}_0, \phi_0)\) satisfying \( \| (\vec{u}_0, \psi_0) \|_{X_0} \leq \varepsilon \), then (29) possesses a unique global solution \((u, v, \phi) \in X\). Moreover, the following decay estimates hold

\[
\| u(t) \|_{L_x^\infty} \lesssim \varepsilon t^{-1}; \quad \| v(t) \|_{L_x^\infty} \lesssim \varepsilon t^{-\frac{3}{2}}; \quad \| \psi(t) \|_{L_x^\infty} \lesssim \varepsilon t^{-\frac{1}{2}}.
\]
The proof of this theorem relies on the standard continuity argument.

**Proposition (T. Tao’s book)**

Suppose that \((\vec{u}, \psi)\) with the initial data \((\vec{u}_0, \psi_0)\), satisfies

\[
\|(\vec{u}, \psi)\|_X \lesssim \|(\vec{u}_0, \psi_0)\|_{X_0} + C(\|\vec{u}, \psi\|_X),
\]

(41)

where \(C(a) \geq Ca^\beta\) for \(a \lesssim 1, \beta > 1\). There exists \(r_0\) such that, if

\[
\|(\vec{u}_0, \psi_0)\|_{X_0} \lesssim r_0,
\]

then \(\|(\vec{u}, \psi)\|_X \lesssim 2r_0\).
Magnetic Diffusion only, $\nu_1 = \nu_2 = 0, \eta_1 = \eta_2 > 0$

The 2D MHD equations with no dissipation:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases}$$

(42)

The global regularity problem remains open.
Weak solutions are global.

**Theorem**

Let $\kappa > 0$. Let $(u_0, b_0) \in H^1$. Then (42) has a global weak solution $(u, b)$ with

$$(u, b) \in L^\infty([0, \infty); H^1).$$

*Remark.* It remains open whether or not two $H^1$-weak solutions must coincide.

*Remark.* It remains open whether or not the $H^1$-weak solution becomes regular when $(u_0, b_0)$ is more regular, say $(u_0, b_0) \in H^2$. 
Global regularity for MHD equation with \((-\Delta)^\beta b\)

Consider

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b, \quad x \in \mathbb{R}^2, \; t > 0, \\
\partial_t b + u \cdot \nabla b + (-\Delta)^\beta b &= b \cdot \nabla u, \quad x \in \mathbb{R}^2, \; t > 0, \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \quad x \in \mathbb{R}^2, \; t > 0, \\
u(x, 0) &= u_0(x), \quad b(x, 0) = b_0(x), \quad x \in \mathbb{R}^2,
\end{aligned}
\] (43)
Theorem (C. Cao, J. Wu and B. Yuan)

Consider (43) with $\beta > 1$. Assume that $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$ and $j_0 = \nabla \times b_0$ satisfying

$$\|\nabla j_0\|_{L^\infty} < \infty.$$

Then (43) has a unique global solution $(u, b)$ satisfying, for any $T > 0$,

$$(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)), \quad \nabla j \in L^1([0, T]; L^\infty(\mathbb{R}^2))$$

where $j = \nabla \times b$. 
The 2D MHD equations with vertical dissipation and horizontal magnetic diffusion

\[
\begin{align*}
  u_t + u \cdot \nabla u &= -\nabla p + \nu u_{yy} + b \cdot \nabla b, \\
  b_t + u \cdot \nabla b &= \eta b_{xx} + b \cdot \nabla u, \\
  \nabla \cdot u &= 0, \quad \nabla \cdot b = 0.
\end{align*}
\]

Theorem

Assume $u_0 \in H^2(\mathbb{R}^2)$ and $b_0 \in H^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then the aforementioned MHD equations have a unique global classical solution $(u, b)$. In addition, $(u, b)$ satisfies

$$(u, b) \in L^\infty([0, \infty); H^2),$$

$$\omega_y \in L^2([0, \infty); H^1), \quad j_x \in L^2([0, \infty); H^1).$$
The 2D MHD equations with horizontal dissipation and vertical magnetic diffusion

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p + \partial_{xx} u + b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b &= \partial_{xx} b + b \cdot \nabla u, \\
\nabla \cdot u &= 0, \\
\nabla \cdot b &= 0,
\end{align*}
\]

(44)

where we have set \( \nu_1 = \eta_1 = 1 \).

The global regularity for this case can also be established, but this case appears to be more difficult.
Theorem

Assume that \((u_0, b_0) \in H^2(\mathbb{R}^2), \nabla \cdot u_0 = 0 \text{ and } \nabla \cdot b_0 = 0\). Then, (44) has a unique global solution \((u, b)\) satisfying, for any \(T > 0\) and \(t \leq T\),

\[
\begin{align*}
    u, b &\in L^{\infty}([0, T]; H^2(\mathbb{R}^2)), \\
    \partial_x u, \partial_x b &\in L^2([0, T]; H^2(\mathbb{R}^2)).
\end{align*}
\]
References


C. Cao, D. Regmi, J. Wu and X. Zheng, Global regularity for the 2D magnetohydrodynamics equations with horizontal dissipation and horizontal magnetic diffusion, preprint.
Consider the 2D fractional MHD equations

\[
\begin{aligned}
    u_t + u \cdot \nabla u + \nu(-\Delta)^\alpha u &= -\nabla p + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b + \eta(-\Delta)^\beta b &= b \cdot \nabla u, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \\
    u(x, 0) &= u_0(x), \quad b(x, 0) = b_0(x). 
\end{aligned}
\]  

(45)

where

\[
(-\Delta)^\alpha f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).
\]

The aim is at the smallest \( \alpha \) and \( \beta \) for which (45) has a global regular solution.
The results we have indicate three cases:

The subcritical case: $\alpha + \beta > 1$;

The critical case: $\alpha + \beta = 1$;

The supercritical case: $\alpha + \beta < 1$. 
The global regularity results we currently have are for subcritical cases.


What we currently have is the global regularity for

1. $\nu = 0$ and $\beta > 1$


1. $\alpha > 0$ and $\beta > 1$

   X. Xu and Z. Ye, preprint

2. $\alpha > 0$ and $\beta = 1$

Open problems:

1) $\nu = 0$ and $\beta = 1$

2) $\alpha > 1$ and $\eta = 0$

3) $1 < \alpha + \beta < 2$, $0 < \beta < 1$
Thank You Very Much!